Approximation of spectra of advection-diffusion problems: a finite element exterior calculus approach

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Motivation: dynamo theory

⟨Arnol'd–Korkina '83. Arnol'd–Khesin '21⟩

Numerical computation of fastest growing mode of magnetic field in dynamo theory

Magnetic induction equation (linear dynamo theory):

$$\partial_t \boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - R_m^{-1}\,\nabla\times\nabla\times\boldsymbol{B} = 0$$

with $\boldsymbol{u}$ fixed velocity field and R_m magnetic Reynolds number

Related to the eigenvalue with the largest real part and pseudospectrum of the advection-diffusion operator

$$\mathcal{L}_{\boldsymbol{u}} : \boldsymbol{B} \mapsto \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + R_m^{-1}\,\nabla\times\nabla\times\boldsymbol{B}$$

Lie derivative and Cartan's magic formula

Vectorfield β on a manifold $\mathcal{M}$
$\Phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ flow generated by β

Definition (Lie derivative of $\omega \in \Lambda^k$ along β)

$$L_\beta \omega = \lim_{\tau \to 0} \frac{\Phi_\tau^* \omega - \omega}{\tau}$$

Remark

For 0-forms (standard functions) its value at x is

$$L_\beta \omega(x) = (\beta \cdot \nabla)\omega(x)$$

the derivative of ω along the direction β

When β is a smooth vectorfield

Theorem (Cartan's magic formula)

$$L_\beta = d\iota_\beta + \iota_\beta d$$

Here ι_β is the contraction operator (interior product) and d the exterior derivative

Proxy representation in 3D (k-th order differential forms)

$$\begin{aligned}
k = 0 : &\quad L_\beta p = \beta \cdot \nabla p \\
k = 1 : &\quad L_\beta q = -\beta \times (\nabla \times q) + \nabla(q \cdot \beta) \\
k = 2 : &\quad L_\beta r = (\nabla \cdot r)\beta + \nabla \times (r \times \beta) \\
k = 3 : &\quad L_\beta s = \nabla \cdot (s\beta)
\end{aligned}$$

Recap on differential forms notation

The smooth (closed) complex valued de Rham complex

$$0 \longrightarrow \Lambda^0_{\mathbb{C}}(\Omega) \xrightarrow{d^0} \Lambda^1_{\mathbb{C}}(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n_{\mathbb{C}}(\Omega) \longrightarrow 0$$

Hodge operator $\star : \mathrm{Alt}^k V \to \mathrm{Alt}^{n-k} V$ $\boxed{\star(\star\omega) = (-1)^{k(n-k)}\omega}$

L^2 inner product $(\omega, \eta) = \int_\Omega \omega \wedge \star\bar{\eta}$

Hilbert and Sobolev spaces

$L^2\Lambda^k(\Omega)$ completion of $\Lambda^k(\Omega)$ in the norm $||\cdot||^2_{L^2} = (\cdot,\cdot)$

$H^s\Lambda^k(\Omega)$ differential forms with coefficients in $H^s(\Omega)$

L^2 de Rham complex without and with boundary conditions

Energy Sobolev spaces

$$H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) : d^k\omega \in L^2\Lambda^{k+1}(\Omega)\}$$
$$\mathring{H}\Lambda^k(\Omega) = \{\omega \in H\Lambda^k(\Omega) : \mathrm{Tr}(\omega) = 0\}$$
$$\|\omega\|_{H\Lambda}^2 = \|\omega\|_{L^2}^2 + \|d^k\omega\|_{L^2}^2$$

$$0 \longrightarrow H\Lambda^0(\Omega) \xrightarrow{d^0} H\Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n(\Omega) \longrightarrow 0$$

$$0 \longrightarrow \mathring{H}\Lambda^0(\Omega) \xrightarrow{d^0} \mathring{H}\Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathring{H}\Lambda^n(\Omega) \longrightarrow 0$$

Rosetta stone of differential forms

Differential form		Proxy representation	
		$n=2$	$n=3$
$k=0$	d_0	∇	∇
	$\mathrm{Tr}_{\partial\Omega}$	$p_{\lvert\partial\Omega}$	$p_{\lvert\partial\Omega}$
	$\mathring{H}\Lambda^0$	$H_0^1(\Omega)$	$H_0^1(\Omega)$
	δ_0	-0	0
$k=1$	d_1	$\nabla\times$	$\nabla\times$
	$\mathrm{Tr}_{\partial\Omega}$	$(\boldsymbol{q}\times\boldsymbol{n})_{\lvert\partial\Omega}$	$(\boldsymbol{q}\times\boldsymbol{n})_{\lvert\partial\Omega}$
	$\mathring{H}\Lambda^1$	$\boldsymbol{H}_0(\mathbf{curl})$	$\boldsymbol{H}_0(\mathbf{curl})$
	δ_1	$\nabla\cdot$	$-\nabla\cdot$
$k=2$	d_2	0	$\nabla\cdot$
	$\mathrm{Tr}_{\partial\Omega}$	0	$(\boldsymbol{r}\cdot\boldsymbol{n})_{\lvert\partial\Omega}$
	$\mathring{H}\Lambda^2$	$L_0^2(\Omega)$	$\boldsymbol{H}_0(\mathrm{div})$
	δ_2	$-\nabla\times$	$\nabla\times$
$k=3$	d_3	$-$	0
	$\mathrm{Tr}_{\partial\Omega}$	$-$	0
	$\mathring{H}\Lambda^2$	$-$	$L_0^2(\Omega)$
	δ_3	$-$	$-\nabla$

The dual complex and integration by parts (Stokes formula)

Coderivative $\star\delta^k = (-1)^k d^{n-k}\star$ leads to dual energy Sobolev spaces

$$H^*\Lambda^k(\Omega) = \star H^{n-k}\Lambda(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) : \delta^k\omega \in L^2\Lambda^{k-1}(\Omega)\}$$

$$\mathring{H}^*\Lambda^k(\Omega) = \star\mathring{H}^{n-k}\Lambda(\Omega) = \{\omega \in H^*\Lambda^k(\Omega) : \mathrm{Tr}(\star\omega) = 0\}$$

$$\|\omega\|^2_{H^*\Lambda} = \|\omega\|^2_{L^2} + \|\delta^k\omega\|^2_{L^2}$$

$$0 \longrightarrow \mathring{H}^*\Lambda^n(\Omega) \xrightarrow{\delta^n} \mathring{H}^*\Lambda^{n-1}(\Omega) \xrightarrow{\delta^{n-1}} \cdots \xrightarrow{\delta^1} \mathring{H}^*\Lambda^0(\Omega) \longrightarrow 0$$

$$\boxed{(d^k v, w) = (v, \delta^{k+1} w) + \int_{\partial\Omega} \mathrm{Tr}\, v \wedge \mathrm{Tr}(\star\bar{w})}$$

Lie derivative and advection-diffusion equation for differential forms

The Lie derivative $L_u : \Lambda^k_{\mathbb{C}}(\Omega) \to \Lambda^k_{\mathbb{C}}(\Omega)$ is defined as

$$L_u \omega = \iota_u^{k+1} d^k \omega + d^{k-1} \iota_u^k \omega$$

using the contraction $\iota_u^k : \Lambda^k_{\mathbb{C}}(\Omega) \to \Lambda^{k-1}_{\mathbb{C}}(\Omega)$

Given diffusion $\varepsilon \geq \varepsilon_0 > 0$ and vectorfield u, the advection-diffusion equation reads
For $f \in \Lambda^k_{\mathbb{C}}(\Omega)$ find a time dependent differential k-form $v(t)$ such that

$$\begin{cases} \partial_t v(t) - \varepsilon d^{k-1} \delta^k v(t) - L_u v(t) = f(t) \\ d^k v(t) = 0 \end{cases}$$

Magnetic advection-diffusion equation

For $n = 3$ and $k = 2$ we get
(scalar advection-diffusion
Fokker–Planck equation)

$$\begin{cases} \partial_t \boldsymbol{B} - \varepsilon \nabla \times \nabla \times \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

Generalized advection-diffusion eigenvalue problem

Find $\lambda \in \mathbb{C}$ and a k-form $v \neq 0$ such that

$$\begin{cases} \varepsilon d^{k-1}\delta^k v + L_u v = \lambda v \\ d^k v = 0 \end{cases}$$

For nonzero eigenvalues λ, this problem is equivalent to

$$\varepsilon d^{k-1}\delta^k v + d^{k-1}\iota_u^k v = \lambda v$$

This comes from $L_u v = \iota_u^{k+1} d^k v + d^{k-1}\iota_u^k v = d^{k-1}\iota_u^k v$
and from the fact that taking d^k of the first equation we have $d^k v = 0$

Mixed variational formulation (with boundary conditions)

Enforcing the constraint $d^k v = 0$ via a Lagrange multiplier $p \in \mathring{H}^* \Lambda^{k+1}(\Omega)$

we get the analogous of the Kikuchi formulation for Maxwell

Find $\lambda \in \mathbb{C}$ and $v \in \mathring{H}^* \Lambda^k(\Omega)$ with $v \neq 0$ such that for some $p \in \mathring{H}^* \Lambda^{k+1}(\Omega)$

$$\begin{cases} \varepsilon(\delta^k v, \delta^k w) + (\iota_u^k v, \delta^k w) + (\delta^{k+1} p, w) = \lambda(v, w) & \forall w \in \mathring{H}^* \Lambda^k(\Omega) \\ (v, \delta^{k+1} q) = 0 & \forall q \in \mathring{H}^* \Lambda^{k+1}(\Omega) \end{cases}$$

Let's go back to the theory of eigenvalues approximation in mixed form in the case of Maxwell (no convection)

We now revert to the language of proxy and describe Maxwell's eigenvalues

Standard variational formulation

$$\lambda \in \mathbb{R},\ \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}),\ \boldsymbol{u} \neq 0:$$

$$\begin{cases} (\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}) & \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl}) \\ (\boldsymbol{u}, \nabla\,\phi) = 0 & \forall\phi\in H_0^1 \end{cases} \qquad \begin{matrix} (\delta^k v, \delta^k w) = \lambda(v,w) \\ (v, \delta^{k+1} q) = 0 \end{matrix}$$

for $\lambda \neq 0$

$$\lambda \in \mathbb{R},\ \lambda \neq 0,\ \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}),\ \boldsymbol{u} \neq 0:$$

$$(\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl}) \qquad (\delta^k v, \delta^k w) = \lambda(v,w)$$

Mixed formulations for Maxwell

⟨Kikuchi '89⟩

Divergence free constraint imposed via Lagrange multiplier p

$$\lambda \in \mathbb{R},\ \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}),\ \boldsymbol{u} \neq 0,\ p \in H^1_0 :$$

$$\begin{cases} (\nabla\times \boldsymbol{u}, \nabla\times \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) = \lambda(\boldsymbol{u}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \quad (\delta^k v, \delta^k w) + (\delta^{k+1} p, w) = \lambda(v, w) \\ (\boldsymbol{u}, \nabla q) = 0 & \forall q \in H^1_0 \quad (v, \delta^{k+1} q) = 0 \end{cases}$$

⟨B–Fernandes–Gastaldi–Perugia '99⟩

Second mixed formulation ($\boldsymbol{H}_0(\mathrm{div}^0) = \nabla\times(\boldsymbol{H}_0(\mathbf{curl}))$)

$$\lambda \in \mathbb{R},\ \boldsymbol{\sigma} \in \boldsymbol{H}_0(\mathbf{curl}),\ \boldsymbol{z} \in \boldsymbol{H}_0(\mathrm{div}^0),\ \boldsymbol{z} \neq 0 :$$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{z}, \nabla\times \boldsymbol{\tau}) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{curl}) \quad (v, w) + (l, \delta^k w) \\ (\nabla\times \boldsymbol{\sigma}, \boldsymbol{w}) = -\lambda(\boldsymbol{z}, \boldsymbol{w}) & \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div}^0) \quad (\delta^k v, m) = -\lambda(l, m) \end{cases}$$

Discrete spaces and notation

Notation

(V_ℓ, Q_ℓ) FE spaces for the Kikuchi formulation $\subset (\boldsymbol{H}_0(\mathbf{curl}), H^1_0)$
(V_ℓ, Z_ℓ) FE spaces for the alternative formulation $\subset (\boldsymbol{H}_0(\mathbf{curl}), \nabla\times \boldsymbol{H}_0(\mathbf{curl}))$

$$K^d_\ell = \{\boldsymbol{v}_\ell \in V_\ell : (\boldsymbol{v}_\ell, \nabla q_\ell) = 0 \ \forall q_\ell \in Q_\ell\}$$
$$K^c_\ell = \{\boldsymbol{\tau}_\ell \in V_\ell : (\nabla\times \boldsymbol{\tau}_\ell, w_\ell) = 0 \ \forall w_\ell \in Z_\ell\}$$

$(V_0, Q_0) \subset (H_0(\mathbf{curl}), H^1_0)$ space of solutions to Kikuchi source problem with $\boldsymbol{f} \in \boldsymbol{L}^2$
$(V^0, Z^0) \subset (H_0(\mathbf{curl}), \nabla\times \boldsymbol{H}_0(\mathbf{curl}))$ solutions to alternative source problem with $\boldsymbol{g} \in \boldsymbol{L}^2$

$$\begin{cases} (\nabla\times \boldsymbol{u}, \nabla\times \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ (\boldsymbol{u}, \nabla q) = 0 & \forall q \in H^1_0 \end{cases}$$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{z}, \nabla\times \boldsymbol{\tau}) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{curl}) \\ (\nabla\times \boldsymbol{\sigma}, \boldsymbol{w}) = -(\boldsymbol{g}, \boldsymbol{w}) & \forall \boldsymbol{w} \in \nabla\times(\boldsymbol{H}_0(\mathbf{curl})) \end{cases}$$

Mixed conditions for Kikuchi formulation

[ELKER] Ellipticity in the discrete kernel

There exists $\alpha > 0$ such that

$$(\nabla\times \boldsymbol{v}_\ell, \nabla\times \boldsymbol{v}_\ell) \geq \alpha\|\boldsymbol{v}_\ell\|^2_{L^2} \quad \forall \boldsymbol{v}_\ell \in K^d_\ell$$

[WA1] Weak approximability of $Q_0 \subset H^{1+s} \cap H^1_0$

There exists $\omega_1(\ell)$ tending to zero such that

$$\sup_{\boldsymbol{v}_\ell \in K^d_\ell} \frac{(\boldsymbol{v}_\ell, \nabla\psi)}{\|\boldsymbol{v}_\ell\|_{\mathbf{curl}}} \leq \omega_1(\ell)\|\psi\|_{H^1} \quad \forall \psi \in Q_0$$

[SA1] Strong approximability of $V_0 \subset \boldsymbol{H}^s_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}^0)$

There exists $\omega_2(\ell)$ tending to zero such that for every $\boldsymbol{u} \in V_0$ there exists $\boldsymbol{u}^I \in K^d_\ell$ such that

$$\|\boldsymbol{u} - \boldsymbol{u}^I\|_{\mathbf{curl}} \leq \omega_2(\ell)\|\boldsymbol{u}\|_{V_0}$$

Mixed conditions for second formulation

[WA2] Weak approximability of $Z^0 = \boldsymbol{H}_0^s(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}^0)$

There exists $\omega_3(\ell)$ tending to zero such that

$$(\mathbf{curl}\, \boldsymbol{\tau}_\ell, \boldsymbol{z}) \leq \omega_3(\ell) \|\boldsymbol{\tau}_\ell\|_{L^2} \|\boldsymbol{z}\|_{Z^0} \quad \forall \boldsymbol{\tau}_\ell \in K_\ell^c,\ \forall \boldsymbol{z} \in Z^0$$

[SA2] Strong approximability of $Z^0 = \boldsymbol{H}_0^s(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}^0)$

There exists $\omega_4(\ell)$ tending to zero such that for every $\boldsymbol{z} \in Z^0$ there exists $\boldsymbol{z}^I \in Z_\ell$ such that

$$\|\boldsymbol{z} - \boldsymbol{z}^I\|_{L^2} \leq \omega_4(\ell) \|\boldsymbol{z}\|_{Z^0}$$

Fortin operator

$\Pi_\ell : V^0 \to V_\ell$ such that $\forall \boldsymbol{\sigma} \in V^0$

$$(\nabla\times(\boldsymbol{\sigma} - \Pi_\ell\boldsymbol{\sigma}), \boldsymbol{w}_\ell) = 0 \quad \forall \boldsymbol{w}_\ell \in Z_\ell$$
$$\|\Pi_\ell\boldsymbol{\sigma}\|_{\mathbf{curl}} \le C\|\boldsymbol{\sigma}\|_{V^0}$$

[FORTID] Fortid property

There exists $\omega_5(\ell)$ tending to zero such that

$$\|\boldsymbol{\sigma} - \Pi_\ell\boldsymbol{\sigma}\|_{\boldsymbol{L}^2} \le \omega_5(\ell)\|\boldsymbol{\sigma}\|_{V^0} \quad \forall \boldsymbol{\sigma} \in V^0$$

Let's see how these conditions can be used for the convergence

Kikuchi solution operators: continuous...

$$\begin{cases} (\nabla\times \boldsymbol{u}, \nabla\times \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ (\nabla q, \boldsymbol{u}) = 0 & \forall q \in H_0^1 \end{cases}$$

$$T^{Ki} \in \mathcal{L}(\boldsymbol{L}^2),\ T^{Ki}(\boldsymbol{f}) = \boldsymbol{u}$$

...and discrete

$$\begin{cases} (\nabla\times \boldsymbol{u}_\ell, \nabla\times \boldsymbol{v}) + (\nabla p_\ell, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) & \forall \boldsymbol{v} \in V_\ell \\ (\nabla q, \boldsymbol{u}_\ell) = 0 & \forall q \in Q_\ell \end{cases}$$

$$T^{Ki}_\ell \in \mathcal{L}(\boldsymbol{L}^2),\ T^{Ki}_\ell(\boldsymbol{f}) = \boldsymbol{u}_\ell$$

Alternative solution operators: continuous...

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\nabla\times \boldsymbol{\tau}, \boldsymbol{z}) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{curl}) \\ (\nabla\times \boldsymbol{\sigma}, \boldsymbol{w}) = -(\boldsymbol{g}, \boldsymbol{w}) & \forall \boldsymbol{w} \in \nabla\times(\boldsymbol{H}_0(\mathbf{curl})) \end{cases}$$

$$T^{M2} \in \mathcal{L}(\boldsymbol{L}^2),\ T^{M2}(\boldsymbol{g}) = \boldsymbol{z}$$

...and discrete

$$\begin{cases} (\boldsymbol{\sigma}_\ell, \boldsymbol{\tau}) + (\nabla\times \boldsymbol{\tau}, \boldsymbol{z}_\ell) = 0 & \forall \boldsymbol{\tau} \in V_\ell \\ (\nabla\times \boldsymbol{\sigma}_\ell, \boldsymbol{w}) = -(\boldsymbol{g}, \boldsymbol{w}) & \forall \boldsymbol{w} \in Z_\ell \end{cases}$$

$$T^{M2}_\ell \in \mathcal{L}(\boldsymbol{L}^2),\ T^{M2}_\ell(\boldsymbol{g}) = \boldsymbol{z}_\ell$$

Convergence in norm

⟨B.–Brezzi–Gastaldi ’97⟩

Theorem

If the ellipticity in the discrete kernel [ELKER], the weak approximability of Q [WA1], and the strong approximability of V_0 [SA1] are satisfied, then the following convergence in norm holds true

$$\|T^{Ki} - T_\ell^{Ki}\|_{\mathcal{L}(\boldsymbol{L}^2),\boldsymbol{H}(\mathbf{curl})} \to 0$$

Theorem

If the weak approximability of Z^0 [WA2] and the strong approximability of Z^0 [SA2] are satisfied, and if there exists a Fortin operator satisfying the Fortid property [FORTID], then the following convergence in norm holds true

$$\|T^{M2} - T_\ell^{M2}\|_{\mathcal{L}(\boldsymbol{L}^2)} \to 0$$

Compactness properties

The space $\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}^0)$ is compactly embedded in L^2

$\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$ and $\nabla \cdot \boldsymbol{v} = 0$

Compactness can be rephrased as

Given a sequence $\{\boldsymbol{u}_n\} \subset \boldsymbol{H}_0(\mathbf{curl})$ such that $\forall n$

$$(\boldsymbol{u}_n, \nabla \phi) = 0 \quad \forall \phi \in H_0^1$$

If $\{\boldsymbol{u}_n\}$ is uniformly bounded in $\boldsymbol{H}_0(\mathbf{curl})$, then there exits a subsequence (still denoted $\{\boldsymbol{u}_n\}$) and $\boldsymbol{u} \in \boldsymbol{L}^2$ such that

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^2} \to 0$$

Remark

It follows $\nabla \cdot \boldsymbol{u} = 0$

Discrete compactness property

Discrete analogue for the spaces $V_\ell \subset \boldsymbol{H}_0(\mathbf{curl})$ and $Q_\ell \subset H^1_0$.

Given a sequence $\{\boldsymbol{u}_n\} \subset V_{\ell_n}$ *discretely divergence free*, i.e., $\forall n$

$$(\boldsymbol{u}_n, \nabla\phi) = 0 \quad \forall \phi \in Q_{\ell_n}$$

If $\{\boldsymbol{u}_n\}$ is uniformly bounded in $\boldsymbol{H}_0(\mathbf{curl})$, then there exits a subsequence (still denoted $\{\boldsymbol{u}_n\}$) and $\boldsymbol{u} \in \boldsymbol{L}^2$ such that

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^2} \to 0$$

[SDCP] Strong DCP

We say that the SDCP is satisfied if $\boldsymbol{u}$ is divergence free $\nabla\cdot\boldsymbol{u} = 0$
This is true, for instance, if Q_ℓ is a good approximation to H^1_0

Equivalence

⟨B. '07,'10⟩

Given $V_\ell \subset \boldsymbol{H}_0(\mathbf{curl})$, construct Q_ℓ and Z_ℓ such that $\nabla Q_\ell \subset V_\ell, \quad \nabla\times V_\ell \subset Z_\ell$

- $Z_\ell = \nabla\times V_\ell$
- The kernel of **curl** in V_ℓ consists of gradient. Take Q_ℓ as the set of potentials vanishing on the boundary $\partial\Omega$

Theorem

The following three sets of conditions are equivalent

i) *ELKER, WA1, SA1*

ii) *WA2, SA2, FORTID*

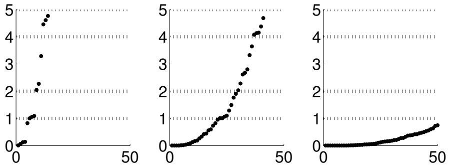
iii) *SDCP and standard approximation property: for any* $\boldsymbol{v} \in V_0$ *there exists* $\boldsymbol{v}_\ell^I \in V_\ell$ *such that*

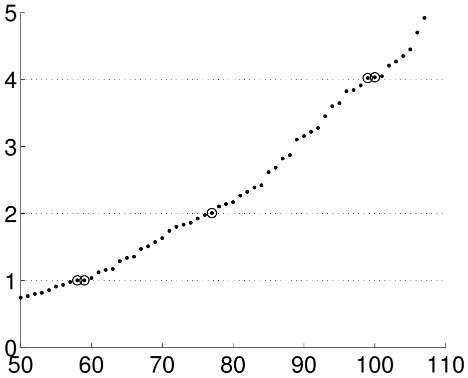
$$\|\boldsymbol{v} - \boldsymbol{v}_\ell^I\|_{\boldsymbol{H}(\mathbf{curl})} \to 0$$

N.B.: The proof relies on V_0 compact in $\boldsymbol{H}(\mathbf{curl})$

Standard P1 finite elements in 2D

Unstructured mesh ($N = 4, 8, 16$)





Zoom for $N = 16$

Standard P1 finite elements on 2D (some eigenfunctions)

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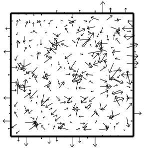

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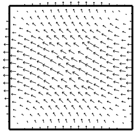

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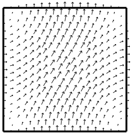

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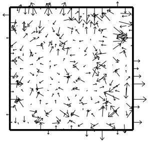

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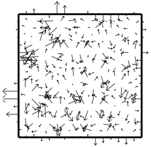

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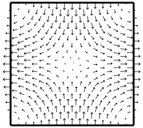

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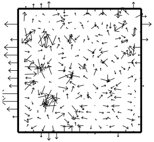
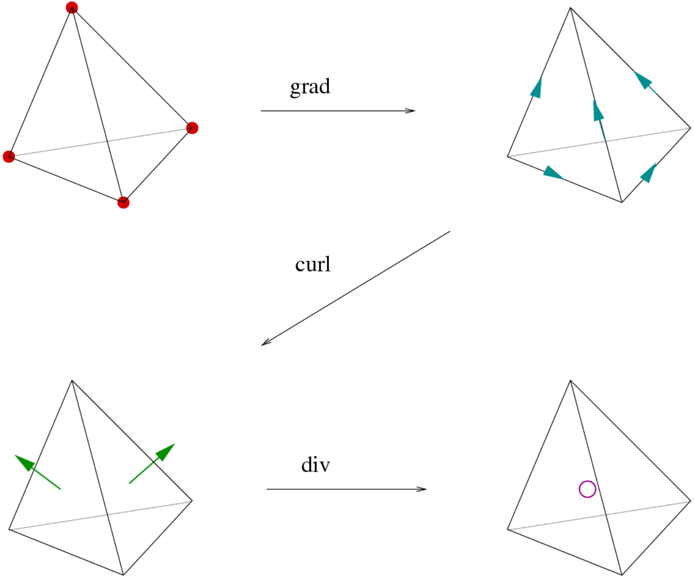


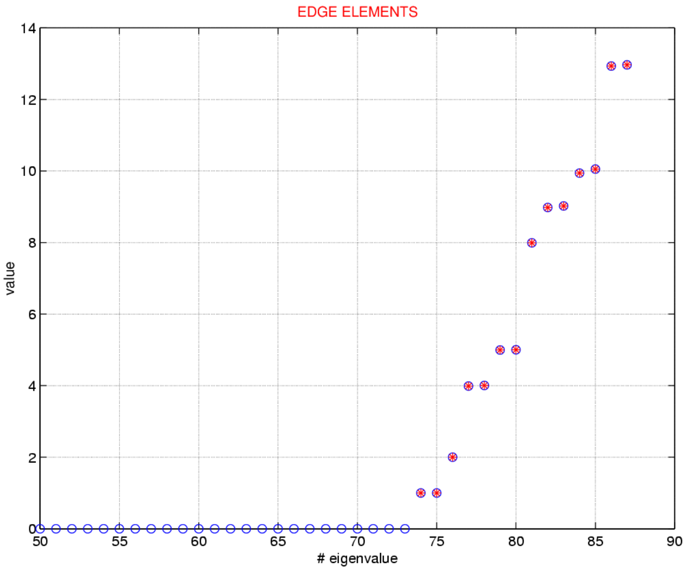
Criss-cross mesh with P1 elements

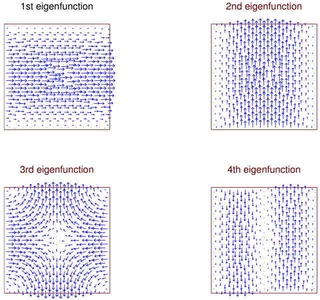
	Computed (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
15		9.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
zero	3	15	63	255	1023
dof	14	62	254	1022	4094

Edge finite elements (Raviart–Thomas–Nédélec)



Computations with edge elements





Back to the framework of differential forms

Let's see how this theory can be extended to our problem

Find $\lambda \in \mathbb{C}$ and $v \in \mathring{H}^*\Lambda^k(\Omega)$ with $v \neq 0$ such that for some $p \in \mathring{H}^*\Lambda^{k+1}(\Omega)$

$$\begin{cases} \varepsilon(\delta^k v, \delta^k w) + (\iota_u^k v, \delta^k w) + (\delta^{k+1} p, w) = \lambda(v, w) & \forall w \in \mathring{H}^*\Lambda^k(\Omega) \\ (v, \delta^{k+1} q) = 0 & \forall q \in \mathring{H}^*\Lambda^{k+1}(\Omega) \end{cases}$$

We allow for non trivial topology, so that we have to account for harmonic forms

$$\mathcal{H}^k = \{\omega \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) : d^k\omega = 0,\ \delta^k\omega = 0\}$$

Shifted bilinear form

We introduce the kernel

$$\mathbb{K}^k = \{v \in \mathring{H}^*\Lambda^k(\Omega) : (v, \delta^{k+1}p) = 0,\ \forall p \in \mathring{H}^*\Lambda^{k+1}(\Omega)\}$$

so that our mixed problem reduces to: find $\lambda \in \mathbb{C}$ and $v \in \mathbb{K}^k$ with $v \neq 0$ such that

$$\varepsilon(\delta^k v, \delta^k w) + (\iota_u^k v, \delta^k w) = \lambda(v, w) \quad \forall w \in \mathbb{K}^k$$

Bilinear form obtained after a shift $\nu > 0$ allows us to define the solution operator

$$a : \mathring{H}^*\Lambda^k(\Omega) \times \mathring{H}^*\Lambda^k(\Omega) \to \mathbb{C}$$
$$a(v, w) = \varepsilon(\delta^k v, \delta^k w) + (\iota_u^k v, \delta^k w) + \nu(v, w)$$

Solution operator

Lemma

Assume that $u \in L^\infty(\Omega)$

Then there exist positive constants $\nu \geq \frac{C\|u\|^2_{L^\infty}}{2\varepsilon} + 1 > 0$ *and* $\alpha = \min\{\frac{\varepsilon}{4}, \frac{1}{2}\} > 0$
such that for any $v \in \Lambda^k_{\mathbb{C}}(\Omega)$ $w \in \Lambda^{k-1}_{\mathbb{C}}(\Omega)$ *and* $0 \leq k \leq n$ *we have*

$$|\varepsilon(w,w) + (\iota^k_u v, w) + \nu(v,v)| \geq \alpha(\|v\|^2_{L^2} + \|w\|^2_{L^2})$$

It follows that our bilinear form is coercive

$$|a(v,v)| = |\varepsilon(\delta^k v, \delta^k v) + (\iota^k_u v, \delta^k v) + \nu(v,v)| \geq \alpha\|v\|^2_{H^*\Lambda} \quad \forall v \in \mathring{H}^*\Lambda^k(\Omega)$$

Solution operator (Lax–Milgram) $T : L^2\Lambda^k(\Omega) \to \mathbb{K}^k$

$$a(Tf, w) = (f, w) \quad \forall w \in \mathbb{K}^k$$

Compactness of solution operator

Ω is said s-regular $(0 \le s \le 1)$ if for any $v \in \mathring{H}^*\Lambda^k(\Omega) \cap H\Lambda^k(\Omega)$

$$\|v\|_{H^s} \le C(\|v\|_{L^2} + \|\delta^k v\|_{L^2} + \|d^k v\|_{L^2})$$

Standard regularity assumption

We assume that the domain Ω is s-regular for some $s \in (1/2, 1]$

$$V_0 = T(L^2\Lambda^k(\Omega)) \subset \mathring{H}^*\Lambda^k(\Omega) \cap H\Lambda^k(\Omega) \hookrightarrow H^s\Lambda^k(\Omega)$$

implies T compact in $\mathcal{L}(L^2)$

Our theory needs a stronger compactness

We want to use T compact in $\mathcal{L}(L^2\Lambda^k(\Omega), \mathring{H}^*\Lambda^k(\Omega))$

For Maxwell it is $\mathcal{L}(\mathbf{L}^2, \boldsymbol{H}(\mathbf{curl}))$ compared to $\mathcal{L}(\mathbf{L}^2, \mathbf{L}^2)$

Compactness of solution operator (cont'd)

Lemma (Regularity)

Assume that the vectorfield $u \in W^{1,\infty}$. Then V_0 is compactly embedded in $\mathring{H}^*\Lambda^k(\Omega)$

Proof.

$v = Tf \in V_0 \subset \mathbb{K}^k \subset H^s\Lambda^k(\Omega)$. We want to show $\delta^k V_0$ compact in $L^2\Lambda^{k-1}(\Omega)$

$$\varepsilon d^{k-1}\delta^k v + d^{k-1}\iota_u^k v + \delta^{k+1}p + \nu v = f$$

$$u \in W^{1,\infty} \implies d^{k-1}\iota_u^k v \in H^{s-1}\Lambda^k \implies d^{k-1}\delta^k v = \tfrac{1}{\varepsilon}(f - d^{k-1}\iota_u^k v - \delta^{k+1}p - \nu v) \in H^{s-1}\Lambda^k$$

There exists $q \in H_0^1\Lambda^k(\Omega)$ with $d^k q = 0$ such that $\delta^k v = \delta^k q$ which gives

$$(d^{k-1}\delta^k + \delta^{k+1}d^k)q = d^{k-1}\delta^k q = d^{k-1}\delta^k v \in H^{s-1}\Lambda^k(\Omega), \quad q|_{\partial\Omega} = 0$$

The result follows then by elliptic regularity of the Hodge–Laplace operator $(t \in (1/2, s))$ $\delta^k v = \delta^k q \in H^t\Lambda^{k-1}(\Omega)$

□

Discretization of the dual complex

⟨Arnold–Falk–Winther '06, '10⟩

We want to construct approximations of the dual complex

$$0 \longrightarrow \mathring{H}^*\Lambda^n(\Omega) \xrightarrow{\delta^n} \mathring{H}^*\Lambda^{n-1}(\Omega) \xrightarrow{\delta^{n-1}} \cdots \xrightarrow{\delta^1} \mathring{H}^*\Lambda^0(\Omega) \longrightarrow 0$$

Given standard FEEC spaces $\widetilde{\Lambda}_h^k \subset \mathring{H}\Lambda^k(\Omega)$ we define

$$\Lambda_h^k = \star\widetilde{\Lambda}_h^{n-k} \subset \mathring{H}^*\Lambda^k$$

and obtain a conforming finite element subcomplex

Discrete harmonic forms

$$\mathcal{H}_h^k = \{u \in \Lambda_h^k : \delta^k u = 0,\ (u, \delta^{k+1}v) = 0\ \forall v \in \Lambda_h^{k+1}\}$$

Lemma

For any $q_h \in \mathcal{H}_h^k$ there exists $r \in \mathcal{H}^k$ such that $\|r\|_{L^2} \le \|q_h\|_{L^2}$ and

$$\|r - q_h\|_{L^2} \le \|(I - \pi_h^k)r\|_{L^2} \le Ch^s\|r\|_{H^s} \le Ch^s\|q_h\|_{L^2}$$

Bounded cochain property

We assume that there exist bounded projections $\tilde{\pi}_h^k : \mathring{H}\Lambda^k(\Omega) \to \widetilde{\Lambda}_h^k$ such that

$$d^k \tilde{\pi}_h^k = \tilde{\pi}_h^{k+1} d^k$$

i.e., the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathring{H}\Lambda^0(\Omega) & \xrightarrow{d^0} & \cdots & \xrightarrow{d^{n-1}} & \mathring{H}\Lambda^n(\Omega) & \longrightarrow & 0 \\ & & \downarrow \tilde{\pi}_h^0 & & \downarrow & & \downarrow \tilde{\pi}_h^n & & \\ 0 & \longrightarrow & \widetilde{\Lambda}_h^0 & \xrightarrow{d^0} & \cdots & \xrightarrow{d^{n-1}} & \widetilde{\Lambda}_h^n & \longrightarrow & 0 \end{array}$$

Moreover, for any $u \in \mathring{H}\Lambda^k(\Omega) \cap H^t\Lambda^k(\Omega)$ with $1/2 < t \leq 1$ we have

$$\|u - \tilde{\pi}_h^k u\|_{L^2} \leq C h^t |u|_{H^t}$$

where the constant $C > 0$ depends on t but can grow unboundedly as $t \to 1/2$

Discrete problem

Find $\lambda_h \in \mathbb{C}$ and $v_h \in \Lambda_h^k$ with $v_h \neq 0$ such that for some $p_h \in \Lambda_h^{k+1}$

$$\begin{cases} \varepsilon(\delta^k v_h, \delta^k w_h) + (\iota_u^k v_h, \delta^k w_h) + (\delta^{k+1} p_h, w_h) = \lambda_h (v_h, w_h) & \forall w_h \in \Lambda_h^k \\ (v_h, \delta^{k+1} q_h) = 0 & \forall q_h \in \Lambda_h^{k+1} \end{cases}$$

The discrete kernel $\quad \mathbb{K}_h^k = \{v_h \in \Lambda_h^k : (v_h, \delta^{k+1} p_h) = 0 \ \forall p_h \in \Lambda_h^{k+1}\} \quad$ gives

Find $\lambda_h \in \mathbb{C}$ and $v_h \in \mathbb{K}_h^k$ with $v_h \neq 0$ such that

$$\varepsilon(\delta^k v_h, \delta^k w_h) + (\iota_u^k v_h, \delta^k w_h) = \lambda_h (v_h, w_h) \quad \forall w_h \in \mathbb{K}_h^k$$

Discrete solution operator

We can use the same shifted bilinear form as for the continuous problem

$T_h : L^2\Lambda^k(\Omega) \to \mathbb{K}_h^k$

$$a(T_h f, w_h) = (f, w_h) \quad \forall w_h \in \mathbb{K}_h^k$$

Convergence theory

We extend to this context the theory presented before for Maxwell

- SDCP and approximation implies
- WA1 and SA1 implies
- Convergence in the norm of $\mathcal{L}(L^2\Lambda^k(\Omega), H\Lambda^k(\Omega))$

Strong discrete compactness property

Lemma (Strong discrete compactness)

For any sequence $\{u_n\}$ *with* $u_n \in \mathbb{K}^k_{h_n}$ *(*$h_n \to 0$*) which is uniformly bounded in* $H^*\Lambda^k$ *there exists a subsequence of* $\{u_n\}$ *converging strongly in* $L^2\Lambda^k(\Omega)$ *to a limit* u *with* $(u, \delta^{k+1} q) = 0$ *for all* $q \in \mathring{H}^*\Lambda^{k+1}(\Omega)$

Proof.

The proof is classical. In the literature it is not so common to find it for general topologies

$$u_n = \tilde{u}_n + p_n \quad \text{orthogonal decomposition in } \mathbb{K}^k_{h_n} \times \mathcal{H}^k_{h_n}$$

$q_n \in \mathcal{H}^k$ such that $\|q_n\|_{L^2} \leq \|p_n\|_{L^2}$ and $\|p_n - q_n\|_{L^2} \leq Ch^s \|p_n\|_{L^2}$

$\mathcal{H}^k$ finite dimensional implies $q_n \to q \in \mathcal{H}^k$ in L^2 (up to a subsequence)

$$\implies \qquad p_n \to q \in \mathcal{H}^k \quad \text{in } L^2$$

The rest of the proof extracts a subsequence from $\tilde{u}_n$ as in the case for simple topology

□

Weak/Strong Approximability and convergence in norm

Lemma (Weak Approximability)

There exists $\omega_1(h)$*, tending to zero as* h *goes to zero, such that for every* $q \in \mathring{H}^*\Lambda^{k+1}(\Omega)$

$$\sup_{v_h \in \mathbb{K}_h^k} \frac{|(v_h, \delta^{k+1} q)|}{\|v_h\|_{H^*\Lambda}} \leq \omega_1(h) \|\delta^{k+1} q\|_{L^2}$$

Lemma (Strong Approximability)

There exists $\omega_2(h)$*, tending to zero as* h *goes to zero, such that for any* $u \in V_0$ *there exists* $u_h \in \mathbb{K}_h^k$ *such that*

$$\|u - u_h\|_{H^*\Lambda} \leq \omega_2(h) \|u\|_{V_0}$$

Theorem

Putting things together we have $\|T - T_h\|_{\mathcal{L}(L^2\Lambda^k(\Omega), \mathring{H}^*\Lambda^k(\Omega))} \to 0$

Convergence result

Babuška–Osborn theory can be used to get the rate of convergence

$$\left|\mu^{-1} - \frac{1}{m}\sum_{i=1}^{m} \mu_{ih}^{-1}\right| \leq Ch^{2r}$$

Assuming $\sup_{\substack{v\in E \\ \|v\|_{L^2}=1}} \|(\tilde{T} - \tilde{T}_h)v\|_{H^*\Lambda} \leq Ch^r$ $\sup_{\substack{v\in E^* \\ \|v\|_{L^2}=1}} \|(\tilde{T}^* - \tilde{T}_h^*)v\|_{H^*\Lambda} \leq Ch^r$

And $\sup_{\substack{v\in E^* \\ \|v\|_{L^2}=1}} \|(I - \Pi_{\delta,h}^k)\Pi_\delta^k \tilde{T}^* v\|_{L^2} \leq Ch^r$

Π_δ^k and $\Pi_{\delta,h}^k$ are the L^2 projections from $L^2\Lambda^k(\Omega)$ to $\delta^{k+1}\mathring{H}^*\Lambda^{k+1}(\Omega)$ and $\delta^{k+1}\Lambda_h^{k+1}(\Omega)$

Numerical results

We are investigating the following results

Theorem (V. I. Arnold)

The number of linearly independent stationary k-forms is not less than the k-th Betti number of the manifold $\mathcal{M}$

Theorem (V. I. Arnold)

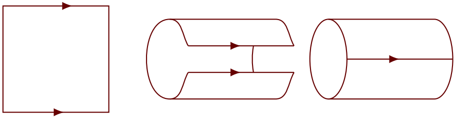
If the diffusion coefficient R_m^{-1} is sufficiently large then the number of linearly independent stationary k-forms is equal to the k-th Betti number of the manifold $\mathcal{M}$

Theorem (C. King)

If the velocity field $\boldsymbol{u}$ is a potential field, i.e. $\boldsymbol{u} = \nabla\,\phi$ for some scalar function ϕ, then the spectrum of the dynamo operator acting on k-forms is real and non-negative

Numerical results: a square with periodic boundary conditions

This resembles a cylindrical surface with Betti numbers $b_0 = 1$, $b_1 = 1$, $b_2 = 0$



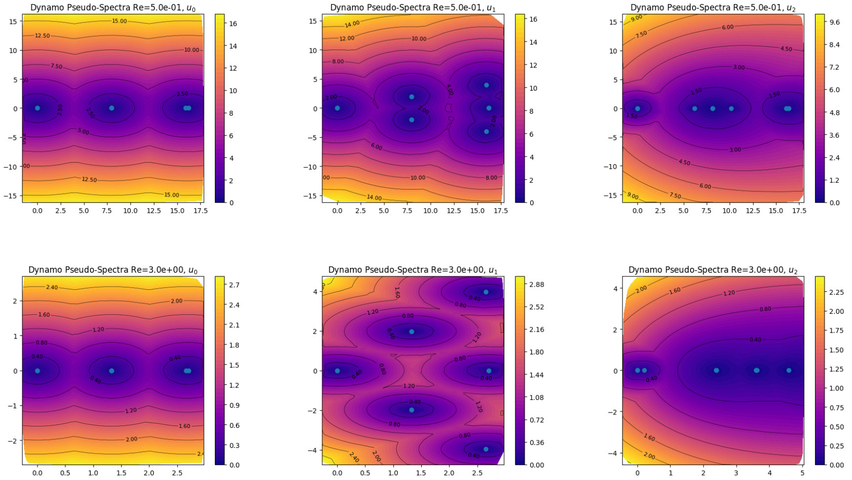
Number of vanishing eigenvalues with $\boldsymbol{u} = (1, 1)$

R_m	100	10	1	0.1
$\dim(\sigma_0)$	1	1	1	1

Spectrum and pseudospectrum

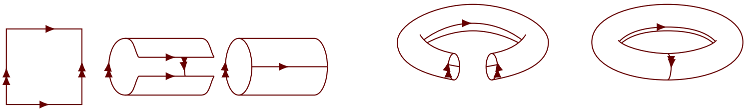
$R_m = 0.5$

$R_m = 3$



$\boldsymbol{u}_0 = (0, 0)$ $\boldsymbol{u}_1 = (1, 1)$ $\boldsymbol{u}_2 = \nabla \sin(\pi x) \sin(\pi y)$

Square domain with fully periodic boundary conditions

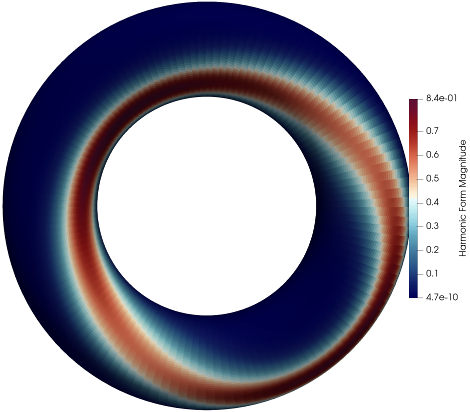


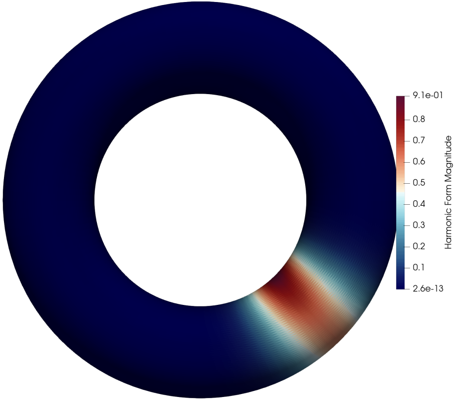
Torus with Betti numbers $b_0 = 1$, $b_1 = 2$, $b_1 = 1$

Number of vanishing eigenvalues with $\boldsymbol{u} = (1, 1, 1)$

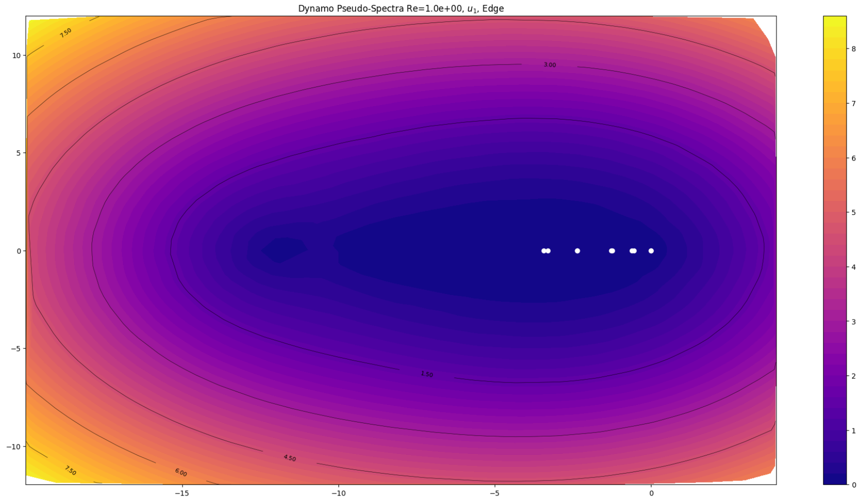
R_m	100	10	1	0.1
$\dim(\sigma_0)$	2	2	2	2

Harmonic forms

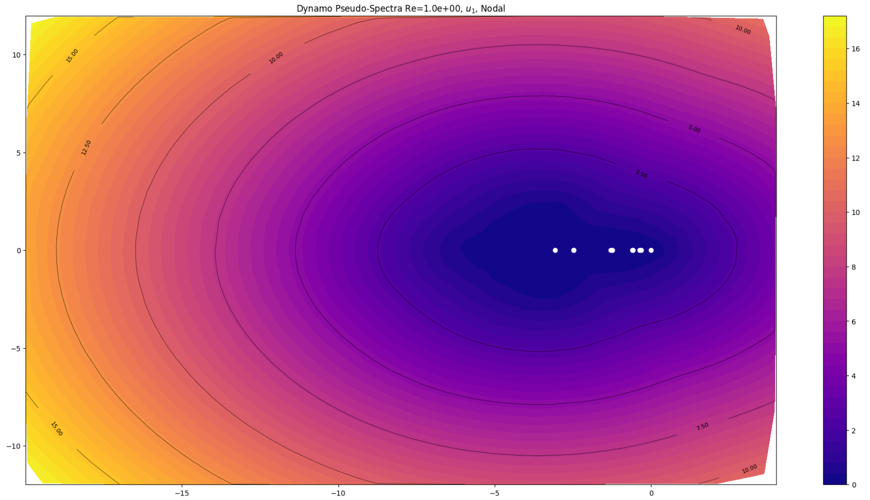




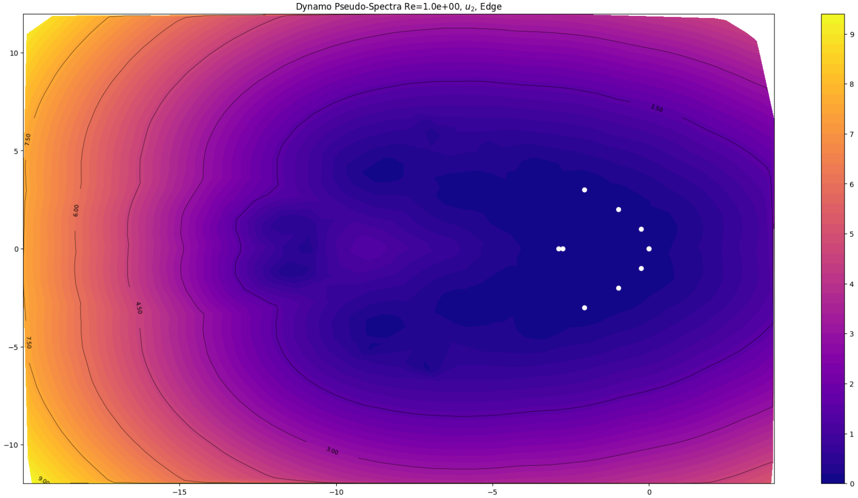
Spectrum and pseudospectrum (edge elements, $\boldsymbol{u} = (1, 1, 1)$)



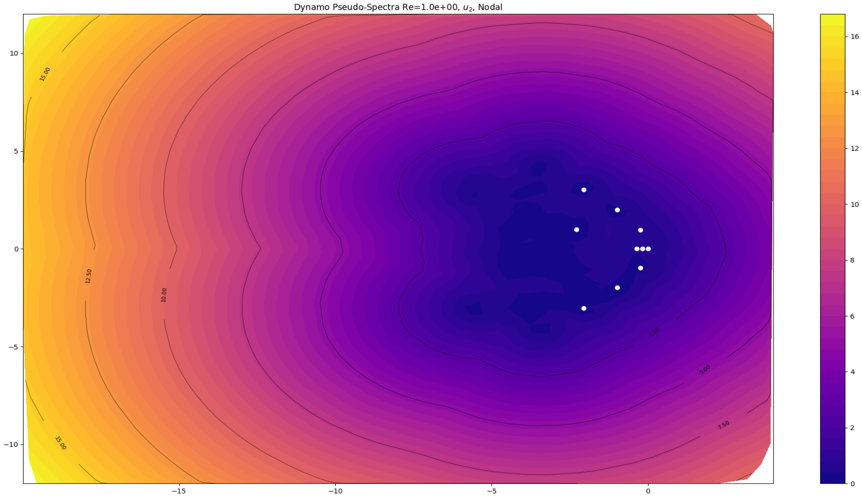
Spectrum and pseudospectrum (nodal elements, $\boldsymbol{u} = (1, 1, 1)$)



Spectrum and pseudospectrum (edge elements, $\boldsymbol{u} = (-y, x, 0)$)



Spectrum and pseudospectrum (nodal elements, $\boldsymbol{u} = (-y, x, 0)$)



Extension to the full Hodge–Laplace equation

So far we have considered the half Hodge–Laplace problem, but we can extend our results to the full one

Find $\lambda \in \mathbb{C}$ and a k-form $v \neq 0$ such that

$$\varepsilon(d^{k-1}\delta^k + \delta^{k+1}d^k)v + L_u v = \lambda v$$

Variational formulation: find $\lambda \in \mathbb{C}$ and $v \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega)$ with $v \neq 0$ such that

$$\varepsilon(\delta^k v, \delta^k w) + \varepsilon(d^k v, d^k w) + (\iota_u^{k+1} d^k v, w) + (\iota_u^k v, \delta^k w) = \lambda(v, w) \quad \forall w \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega)$$

Clearly, this is not amenable to a conforming discretization in $H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega)$

Kikuchi-like formulation

$$\boxed{p = d^k v}$$

Find $\lambda \in \mathbb{C}$ and $v \in \mathring{H}^* \Lambda^k(\Omega)$, $v \neq 0$, such that for some $p \in \mathring{H}^* \Lambda^{k+1}(\Omega)$

$$\begin{cases} \varepsilon(\delta^k v, \delta^k w) + \varepsilon(\delta^{k+1} p, w) + (\iota_u^{k+1} p, w) + (\iota_u^k v, \delta^k w) = \lambda(v, w) & \forall w \in \mathring{H}^* \Lambda^k(\Omega) \\ (p, q) - (v, \delta^{k+1} q) = 0 & \forall q \in \mathring{H}^* \Lambda^{k+1}(\Omega) \end{cases}$$

Discrete version: find $\lambda_h \in \mathbb{C}$ and $v_h \in \Lambda_h^k$, $v_h \neq 0$ such that for some $p_h \in \Lambda_h^{k+1}$

$$\begin{cases} \varepsilon(\delta^k v_h, \delta^k w_h) + \varepsilon(\delta^{k+1} p_h, w_h) + (\iota_u^{k+1} p_h, w_h) + (\iota_u^k v_h, \delta^k w_h) = \lambda_h(v_h, w_h) & \forall w_h \in \Lambda_h^k \\ (p_h, q_h) - (v_h, \delta^{k+1} q_h) = 0 & \forall q_h \in \Lambda_h^{k+1} \end{cases}$$

Convergence in norm follows from inf-sup condition of shifted form

$X = \mathring{H}^*\Lambda^k(\Omega) \times \mathring{H}^*\Lambda^{k+1}(\Omega)$

$A : X \times X \to \mathbb{C}$ for a positive $\nu > 0$

$A((v,p),(w,q)) = \varepsilon(\delta^k v, \delta^k w) + \varepsilon(\delta^{k+1} p, w) + (\iota_u^{k+1} p, w) + (\iota_u^k v, \delta^k w) + (p,q) - (v, \delta^{k+1} q) + \nu(v,w)$

Lemma (Inf-sup conditions)

For $\nu \geq 4\varepsilon + \frac{1}{2}\|u\|_{L^\infty(\Omega)} + \varepsilon^{-1}\|u\|^2_{L^\infty(\Omega)} + 1 > 0$

there exist $\alpha = \min\{\frac{\varepsilon}{4}, \frac{1}{2}\} > 0$ *and* $C > 0$ *such that* $\forall (v,p) \in X \quad \exists (w,q) \in X:$

$$\begin{cases} |A((v,p),(w,q))| \geq \alpha(\|v\|_{H^*\Lambda} + \|p\|_{H^*\Lambda})^2 \\ (\|w\|_{H^*\Lambda} + \|q\|_{H^*\Lambda}) \leq C(\|v\|_{H^*\Lambda} + \|p\|_{H^*\Lambda}) \end{cases}$$

and $\forall (w,q) \in X$ *with* $(w,q) \neq 0 \quad \exists (v,p) \in X$ *such that*

$$|A((v,p),(w,q))| > 0$$

Theorem

Assume that $\nu \geq 4\varepsilon + \frac{1}{2}\|u\|_{L^\infty} + \varepsilon^{-1}\|u\|^2_{L^\infty} + 1 > 0$ *and* u *is a smooth vectorfield. Then we have*

$$\|T - T_h\|_{\mathcal{L}(L^2\Lambda^k(\Omega);L^2\Lambda^k(\Omega))} \to 0 \quad \text{as } h \to 0$$

Theorem

There exists constant $C > 0$ *and* $h_0 > 0$ *such that, for* $0 < h < h_0$,

$$|\lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{i,h}| \leq C\delta_h\delta_h^* \qquad \text{and} \qquad |\lambda - \lambda_{i,h}| \leq C(\delta_h\delta_h^*)^{1/\alpha} \quad i = 1, 2, \cdots, m$$

with

$$\delta_h = \sup_{\substack{U\in E\\ \|U\|_V=1}} \inf_{U_h\in V_h} \|U - U_h\|_V \qquad \text{and} \qquad \delta_h^* = \sup_{\substack{U\in E^*\\ \|U\|_V=1}} \inf_{U_h\in V_h} \|U - U_h\|_V$$

Conclusions

- The convergence theory of eigenvalue problems can be extended from Maxwell to a general advection-diffusion problem for differential forms
- The compactness proof requires extra regularity on the velocity vectorfield