Structure-preserving FEEC methods for spectral and pseudospectral analysis of dynamo operators

UNIVERSITY OF OXFORD
Mathematical Institute

Umberto Zerbinati*, joint work with: Daniele Boffi†, Kaibo Hu*, Yizhou Liang*, Stefano Zampini†

**Mathematical Institute – University of Oxford*

† King Abdullah University of Science and Technology (KAUST)

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Oxford
Mathematics

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Eigenvalue problems are notoriously difficult to solve numerically, as they may exhibit **spurious eigenmodes** in the computed spectrum (among other problems).

We will show how **structure-preserving** methods can be used to avoid spurious eigenmodes in the computed spectrum of dynamo operators.

CHALLENGES IN COMPUTING EIGENVALUES

1

THE LAPLACE EIGENVALUE PROBLEM

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Laplace eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

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The standard variational formulation: find $u \in V := H^1_0(\Omega)$ and $\lambda \in \mathbb{C}$ such that

$$a(u, v) = \lambda m(u, v) \quad \forall\, v \in V,$$

where the bilinear forms are defined as

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx, \quad m(u, v) := \int_\Omega uv \, dx.$$

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Non-linearity of eigenvalue problems

Eigenvalue problems are non-linear, due to the implicit constraint $\|u\|_{L^2(\Omega)} = 1$.

THE LAPLACE EIGENVALUE PROBLEM

We consider a conforming finite element discretization of the variational formulation:

Discrete weak Laplace eigenvalue problem

Find $u_h \in V_h \subset V$ and $\lambda_h \in \mathbb{C}$ such that $\|u_h\|_{L^2(\Omega)} = 1$ and

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Algebraic eigenvalue problem

Find $\mathbf{u} \in \mathbb{C}^N$ and $\lambda_h \in \mathbb{C}$ such that

$$\underline{\underline{\mathbf{A}}}\mathbf{u} = \lambda_h \underline{\underline{\mathbf{M}}}\mathbf{u},$$

where $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{M}}}$ are the stiffness and mass matrices, respectively.

THE LAPLACE EIGENVALUE PROBLEM: NUMERICAL RESULTS

Adopting the obvious strategy works!

$N = 8$	$N = 16$	$N = 32$	$\dots$	$N = \infty$
2.07764608	2.01930990	2.00482122	$\dots$	2
5.33251285	5.08291766	5.02072060	$\dots$	5
5.53254919	5.13018295	5.03235583	$\dots$	5
9.18255754	8.30543350	8.07692593	$\dots$	8
11.6879356	10.3814080	10.0949216	$\dots$	10
11.8419204	10.3900040	10.0954511	$\dots$	10
11.6879356	13.5716234	13.1442958	$\dots$	13
15.2270501	13.9825316	13.2432114	$\dots$	13
17.0125136	18.0416423	17.2561757	$\dots$	17
21.3374450	18.0704980	17.2626019	$\dots$	17

ANOTHER LAPLACE EIGENVALUE PROBLEM

Let us now consider a mixed formulation to preserve the structural distinction between

constitutive relations and **conservation laws**.

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constitutive relations and **conservation laws**.

We thus consider the following Laplace eigenvalue problem with Neumann boundary conditions, in a mixed formulation:

Neumann mixed Laplace eigenvalue problem

Find $u \in L^2(\Omega)$ and $\lambda \in \mathbb{C}$ such that $\|u\|_{L^2(\Omega)} = 1$ and for some $\sigma \in H_0(\mathrm{div};\Omega)$

$$\begin{aligned} (\sigma,\tau)_{L^2} + b(\tau,u) &= 0 && \forall\, \tau \in H_0(\mathrm{div};\Omega), \\ b(\sigma,v) &= -\lambda m(u,v) && \forall\, v \in L^2(\Omega), \end{aligned}$$

where $b(\tau,v) := \int_\Omega \mathrm{div}\,\tau\, v\, dx$ and $(\sigma,\tau)_{L^2} := \int_\Omega \sigma\cdot\tau\, dx$.

ANOTHER LAPLACE EIGENVALUE PROBLEM

Boffi–Brezzi–Gastaldi, Math. Comp., 69 (2000).

Discrete Neumann mixed Laplace eigenvalue problem

Find $u_h \in V_h$ and $\lambda_h \in \mathbb{C}$ such that $\|u_h\|_{L^2(\Omega)} = 1$ and for some $\sigma_h \in \Sigma_h$

$$
\begin{aligned}
(\sigma_h, \tau_h)_{L^2} + b(\tau_h, u_h) &= 0 && \forall\, \tau_h \in \Sigma_h, \\
b(\sigma_h, v_h) &= -\lambda_h m(u_h, v_h) && \forall\, v_h \in V_h.
\end{aligned}
$$

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The ‘inf-sup condition’ is necessary and sufficient for well-posedness of the equation.

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The ‘inf-sup condition’ is necessary and sufficient for well-posedness of the equation.

$\mathbf{Q}_1 - P_0$

The choice of space pair $(\Sigma_h, V_h) = (\mathbf{Q}_1, P_0)$ is **inf-sup stable**, yet it leads to the presence of **spurious eigenvalues** in the computed spectrum!

ANOTHER LAPLACE EIGENVALUE PROBLEM: NUMERICAL RESULTS

Oxford Mathematical Institute

Adopting the obvious strategy does not work!

$N = 8$	$N = 16$	$N = 32$	$\dots$	$N = \infty$
1.01291606	1.00321689	1.00080347	$\dots$	1
1.01291606	1.00321689	1.00080347	$\dots$	1
1.99946179	1.99996684	1.99999796	$\dots$	2
4.20954747	4.05166420	4.01286752	$\dots$	4
4.20954747	4.05166420	4.01286752	$\dots$	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
19.4536673	17.1061707	17.1814052	$\dots$	17
19.4536673	17.7328645	17.1814052	$\dots$	17
19.9601253	17.7328645	17.7706910	$\dots$	18
19.9601253	**17.9749153**	**17.9984795**	$\dots$	**18**

ANOTHER LAPLACE EIGENVALUE PROBLEM: NUMERICAL RESULTS

Oxford Mathematical Institute

$\mathbf{P}_1 - \mathrm{div}(\mathbf{P}_1)$

The choice of space pair $(\Sigma_h, V_h) = (\mathbf{P}_1, \mathrm{div}(\mathbf{P}_1))$ is **inf-sup stable** (on a criss-cross mesh), yet it leads to the presence of **spurious eigenvalues** in the computed spectrum!

$N = 8$	$N = 16$	$N = 32$	...	$N = \infty$
1.00427624	1.00107048	1.00026772	...	1
1.00427624	1.00107048	1.00026772	...	1
2.01711347	2.00428269	2.00107089	...	2
4.06803872	4.01710491	4.00428186	...	4
4.06803872	4.01710491	4.00428186	...	4
5.10634250	5.02673922	5.00669135	...	5
5.10634250	5.02673922	5.00669135	...	5
5.92293444	**5.98074278**	**5.99518232**	**...**	**6**

AN EQUIVALENT PROBLEM IN ELECTROMAGNETICS

Mathematical Institute

Maxwell eigenvalue problem

Find $\mathbf{E} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ and $\omega \in \mathbb{C}$ such that $\|\mathbf{E}\|_{L^2(\Omega)} = 1$ and

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \omega^2 \mathbf{E} && \text{in } \Omega, \\ \nabla \cdot \mathbf{E} &= 0 && \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Weak Maxwell eigenvalue problem

Find $\mathbf{E} \in H_0(\text{curl};\Omega)$ and $\omega \in \mathbb{C}\backslash\{0\}$ such that $\|\mathbf{E}\|_{L^2(\Omega)} = 1$ and

$$c(\mathbf{E},\mathbf{F}) := (\nabla \times \mathbf{E}, \nabla \times \mathbf{F})_{L^2} = \omega^2 m(\mathbf{E},\mathbf{F})_{L^2} \quad \forall\, \mathbf{F} \in H_0(\text{curl};\Omega).$$

AN EQUIVALENT PROBLEM IN ELECTROMAGNETICS

Boffi–Brezzi–Fortin, Mixed Finite Element Methods and Applications, Springer Ser. Comput. Math., 44 (2013).

We introduce a potential variable $\psi \in H^1_0(\Omega)$ and observe that $\nabla\psi \in H_0(\mathrm{curl};\Omega)$ satisfies, in a distributional sense, the identity

$$\hat{\nabla} \times \bar{\nabla} \times \psi = \Delta\psi,$$

where $\hat{\nabla}\times$ and $\bar{\nabla}\times$ are the scalar and vectorial curl operators in 2D, i.e.

$$\hat{\nabla} \times \mathbf{E} = \partial_x \mathbf{E}_y - \partial_y \mathbf{E}_x, \quad \bar{\nabla} \times \psi = \begin{pmatrix} \partial_x\psi \\ -\partial_y\psi \end{pmatrix}.$$

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Equivalence with the mixed Laplace eigenvalue problem

Let $\mathbf{E} = \bar{\nabla} \times \psi$. The potential ψ solves the mixed weak Laplace eigenvalue problem iff $\mathbf{E}$ solves the weak Maxwell eigenvalue problem and $\sigma = \nabla\psi$ is a rotation of $\mathbf{E}$.

MAXWELL EIGENVALUE PROBLEM: NUMERICAL RESULTS P_1 ELEMENTS

Oxford Mathematical Institute

Adopting the obvious strategy does not work!

$N=8$	$N=16$	$N=32$	$\dots$	$N=\infty$
$-4\cdot 10^{-16}$	$-3\cdot 10^{-15}$	$-1\cdot 10^{-15}$	$\dots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$
1.00427624	1.00107044	1.00026768	$\dots$	1
1.00427624	1.00107044	1.00026768	$\dots$	1
2.01711339	2.00428261	2.00107081	$\dots$	2
4.06803863	4.01710482	4.00428176	$\dots$	4
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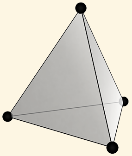
FINITE ELEMENT EXTERIOR CALCULUS (FEEC): EDGE ELEMENTS

Hiptmair, Acta Numer., 11 (2002).

Lagrange nodal elements

Using the space $\mathbf{P}_1$ of Lagrange nodal elements leads to **spurious eigenvalues** in the computed spectrum for the Maxwell eigenvalue problem.



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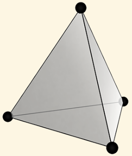
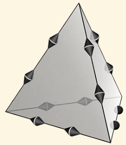


Nédélec edge elements

Using the space $\mathbf{Ned}_1$ of Nédélec edge elements leads to **no spurious eigenvalues** in the computed spectrum for the Maxwell eigenvalue problem.



MAXWELL EIGENVALUE PROBLEM: NUMERICAL RESULTS Ned_1 EDGE ELEMENTS

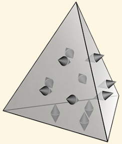
$N=8$	$N=16$	$N=32$	$\dots$	$N=\infty$
$-1\cdot 10^{-15}$	$-1\cdot 10^{-15}$	$4\cdot 10^{-14}$	$\dots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$
1.00106352	1.00026725	1.00006688	$\dots$	1
1.00106352	1.00026725	1.00006688	$\dots$	1
1.99141760	1.99785719	1.99946443	$\dots$	2
4.01665445	4.00425409	4.00106901	$\dots$	4
4.01665445	4.00425409	4.00106901	$\dots$	4
4.97487849	4.99381206	4.99845856	$\dots$	5
4.97487849	4.99381206	4.99845856	$\dots$	5
7.86190171	**7.96567040**	**7.99142874**	$\dots$	**8**

MIXED LAPLACE EIGENVALUE PROBLEM: NUMERICAL RESULTS

Mathematical Institute

$\mathbf{RT_1} - \mathbf{P_0}(\mathcal{T}_h)$

The choice of spaces $\Sigma_h = \mathbf{RT_1}$ and $V_h = P_0$ leads to **no spurious eigenvalues** in the computed spectrum.



$N = 8$	$N = 16$	$N = 32$	...	$N = \infty$
1.00106356	1.00026729	1.00006692	...	1
1.00106356	1.00026729	1.00006692	...	1
1.99141768	1.99785726	1.99946451	...	2
4.01665455	4.00425418	4.00106910	...	4
4.01665455	4.00425418	4.00106910	...	4
4.97487864	4.99381221	4.99845871	...	5
4.97487864	4.99381221	4.99845871	...	5
7.86190193	**7.96567063**	**7.99142897**	**...**	**8**

THE DYNAMO EIGENVALUE PROBLEM

2

DYNAMO EIGENVALUE PROBLEM

Arnold–Khesin, Topological Methods in Hydrodynamics, Appl. Math. Sci. (Springer), 125 (1998), Chap V.

Dynamo eigenvalue problem

Find divergence-free $\mathbf{B}$ and $\lambda \in \mathbb{C}$ such that $\|\mathbf{B}\|_{L^2(\Omega)} = 1$ and

$$\nabla \times (\mathbf{u} \times \mathbf{B}) + R_m^{-1} \nabla \times \nabla \times \mathbf{B} = \lambda \mathbf{B} \quad \text{in } \Omega,$$

where $\mathbf{u}$ is a given divergence-free vector field and R_m is the magnetic Reynolds number.

A COMPUTATIONAL APPROACH TO THE DYNAMO EIGENVALUE PROBLEM

Mathematical Institute

Arnold–Korkina, Moscow Univ. Math. Bull., 38 (1983).

> *... It is* **still unknown** *whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as* $R_m \to \infty$*.*
>
> ...
>
> *Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers* R_m *are of order of magnitude* 10^8*.* **The corresponding matrices are (and will remain) beyond the reach of any computer.**

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... It is **still unknown** *whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as $R_m \to \infty$.*

...

Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8. **The corresponding matrices are (and will remain) beyond the reach of any computer.**

We are not aware of any rigorous study of convergence of the Galerkin method used by Arnold–Korkina for the dynamo eigenvalue problem.
Are there spurious eigenmodes in the computed spectrum?

DYNAMO EIGENVALUE PROBLEM: KIKUCHI'S FORMULATION

Boffi–Hu–Liang–Z, manuscript in preparation.

Dynamo eigenvalue problem: Kikuchi's formulation

Find $\mathbf{B} \in H_0(\mathrm{curl}; \Omega)$, $\psi \in H_0^1(\Omega)$ and $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} c(\mathbf{B}, \mathbf{D}) + (u \times \mathbf{B}, \nabla \times \mathbf{D})_{L^2} + d(\psi, \mathbf{D}) &= \lambda m(\mathbf{B}, \mathbf{D}) && \forall\, \mathbf{D} \in H_0(\mathrm{curl}; \Omega), \\ d(\phi, \mathbf{B}) &= 0 && \forall\, \phi \in H_0^1(\Omega), \end{aligned}$$

where $d(\phi, \mathbf{D}) := \int_\Omega \nabla\phi \cdot \mathbf{D}\, dx$.

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where $d(\phi,\mathbf{D}) := \int_\Omega \nabla\phi \cdot \mathbf{D}\, dx$.

This formulation filters zero eigenvalues associated with the divergence constraint.

DYNAMO EIGENVALUE PROBLEM: DISCRETE KIKUCHI'S FORMULATION

Boffi–Hu–Liang–Z, manuscript in preparation.

Discrete dynamo eigenvalue problem

Find $\mathbf{B}_h \in V_h \subset H_0(\text{curl};\Omega)$, $\psi_h \in W_h \subset H_0^1(\Omega)$ and $\lambda_h \in \mathbb{C}$ such that

$$\begin{aligned} c(\mathbf{B}_h, \mathbf{D}_h) + (\mathbf{u} \times \mathbf{B}_h, \nabla \times \mathbf{D}_h)_{L^2(\Omega)} + d(\psi_h, \mathbf{D}_h) &= \lambda_h m(\mathbf{B}_h, \mathbf{D}_h) && \forall\, \mathbf{D}_h \in \mathbf{V}_h, \\ d(\phi_h, \mathbf{B}_h) &= 0 && \forall\, \phi_h \in Q_h. \end{aligned}$$

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$$\begin{aligned} c(\mathbf{B}_h, \mathbf{D}_h) + (\mathbf{u} \times \mathbf{B}_h, \nabla \times \mathbf{D}_h)_{L^2(\Omega)} + d(\psi_h, \mathbf{D}_h) &= \lambda_h m(\mathbf{B}_h, \mathbf{D}_h) && \forall\, \mathbf{D}_h \in \mathbf{V}_h, \\ d(\phi_h, \mathbf{B}_h) &= 0 && \forall\, \phi_h \in Q_h. \end{aligned}$$

Corollary of the main theorem: $\mathbf{Ned}_1$-P_1 works

The edge element discretization $\mathbf{V}_h = \mathbf{Ned}_1$ and $Q_h = P_1$ is convergent: no spurious or neglected eigenmodes.

DYNAMO EIGENVALUE PROBLEM: HODGE THEORY

Theorem (V. I. Arnold)

The number of zero eigenvalues is **not less than** the first Betti number of Ω.

If the diffusion coefficient R_m^{-1} is sufficiently large, then the number of zero eigenvalues is **equal to** the first Betti number of Ω.

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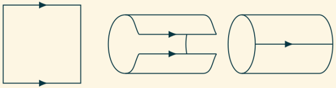

$$b_0 = 1, b_1 = 1, b_2 = 0, \mathbf{u}_1 = (1,1)$$

R_m	100	10	1	0.1
dim(λ_0)	1	1	1	1

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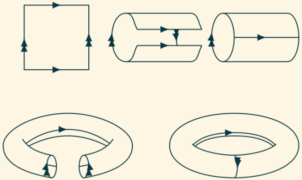


$$b_0 = 1, b_1 = 2, b_2 = 1, \mathbf{u}_1 = (1,1)$$

R_m	100	10	1	0.1
dim(λ_0)	2	2	2	2

DYNAMO EIGENVALUE PROBLEM: WITTEN TRANSFORM

Arnold–Khesin, Topological Methods in Hydrodynamics, Appl. Math. Sci. (Springer), 125 (1998), Rmk V.3.15,
Witten, J. Diff. Geom., 17 (1982).

Theorem (C. King)

Let $\underline{u}$ be a smooth vector field such that there exists a smooth function $\phi : \Omega \to \mathbb{R}$ satisfying $\underline{u} = \nabla\phi$. Then the eigenvalues of the dynamo eigenvalue problem **are all real**.

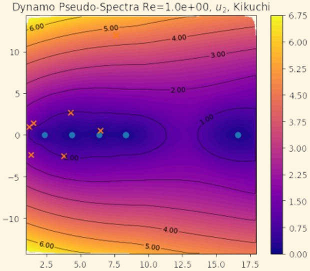


Figure: $\mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y))$, $\times$: $\mathbf{P}_1$, $\circ$: $\mathbf{Ned}_1$.

DYNAMO EIGENVALUE PROBLEM: PSEUDOSPECTRA

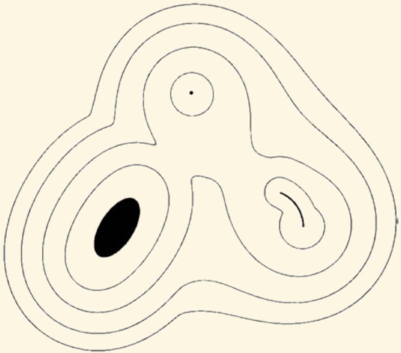
Trefethen–Embree, Spectra and Pseudospectra, Princeton Univ. Press (2005), Thm. 15.3.

Eigenvalue analysis is often **misleading** for assessing instability for **non-normal operators** $(AA^* \neq A^*A)$.

Definition (Pseudospectrum)

Let $A : X \to X$ be a linear operator on a Banach space X. For any $\epsilon > 0$, the ϵ-pseudospectrum of A is defined as

$$\sigma_\epsilon(A) := \{\lambda \in \mathbb{C} : \|(\lambda I - A)^{-1}\|_{L(X,X)} > \epsilon^{-1}\}.$$



DYNAMO EIGENVALUE PROBLEM: PSEUDOSPECTRA

Boffi–Hu–Liang–Z, manuscript in preparation.

Main Theorem (Essentially Kato's first stability estimate)

Let $T : X \to X$ be a compact linear operator on a Hilbert space X and consider a sequence of finite-rank operators $T_n : X \to X$ such that

$$\lim_{\mathbf{n}\to\infty} \|\mathbf{T_n} - \mathbf{T}\|_{\mathbf{L(X,X)}} = \mathbf{0}.$$

Then for any $\epsilon > \delta > 0$, $\exists N > 0$ such that for all $n > N$ we have $\sigma_\delta(T_n) \subset \sigma_\epsilon(T)$.

This is why edge element discretizations of the dynamo eigenvalue problem provide **convergent** approximations of the pseudospectra of the underlying compact operator.

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This is why edge element discretizations of the dynamo eigenvalue problem provide **convergent** approximations of the pseudospectra of the underlying compact operator. There will be **no spurious or neglected eigenmodes** in the spectrum.

DYNAMOS AND EIGENVALUES

3

DYNAMO PROBLEMS IN MAGNETOHYDRODYNAMICS

This dynamo problem arises in the generation of magnetic fields in astrophysical objects.

DYNAMO PROBLEMS IN MAGNETOHYDRODYNAMICS

Mathematical Institute, University of Oxford

This dynamo problem arises in the generation of magnetic fields in astrophysical objects.

Kinematic Dynamo

A field $\mathbf{u}$ is a **kinematic dynamo** if the magnetic energy $\|\mathbf{B}\|_{L^2}$ grows exponentially in time, when $\mathbf{B}$ solves the **magnetic advection-diffusion** equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + R_m^{-1} \nabla \times \nabla \times \mathbf{B},$$

where R_m is the magnetic Reynolds number and $\mathbf{B}$ is the magnetic field. By Gauss's law for magnetism we have

$$\nabla \cdot \mathbf{B} = 0.$$

DYNAMO PROBLEMS IN MAGNETOHYDRODYNAMICS

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$$\nabla \cdot \mathbf{B} = 0.$$

Does there exist a divergence-free **fast kinematic dynamo $\mathbf{u}$** on a given domain Ω?

CASTING AS AN EIGENVALUE PROBLEM

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However, his construction was not quite right.

We propose an alternative construction that does work.

A CLASSICAL RESULT IN SEMIGROUP THEORY

Mathematical Institute

Trefethen–Embree, Spectra and Pseudospectra, Princeton Univ. Press (2005), Thm. 15.3.

Growth bound and spectral bound

Let $L : X \to X$ be a bounded linear operator on a Banach space X that generates a C_0-semigroup, denoted by e^{tL}. Then the following identity holds:

$$\limsup_{t\to\infty} \frac{1}{t} \log\|e^{tL}\|_{L(X,X)} = \alpha(L) := \sup\left\{\mathrm{Re}(\lambda) \, : \, \lambda \in \sigma(L)\right\},$$

where $\alpha(L)$ is called the **spectral abscissa** of L and $\sigma(L)$ is the spectrum of L, i.e.

$$\sigma(L) := \{\lambda \in \mathbb{C} \, : \, (\lambda I - L) \text{ is not bijective}\}.$$

A CLASSICAL RESULT IN SEMIGROUP THEORY—ARNOLD'S KINEMATIC DYNAMO

Mathematical Institute

Arnold–Khesin, Topological Methods in Hydrodynamics, Appl. Math. Sci. (Springer), 125 (1998), Def. V.1.3.

Arnold's definition of a fast kinematic dynamo

Consider the linear operator $L_{\mathbf{u}} : H^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$ defined as

$$L_{\mathbf{u}}(\mathbf{B}) \coloneqq \nabla \times (\mathbf{u} \times \mathbf{B}) + R_m^{-1} \nabla \times \nabla \times \mathbf{B},$$

where $\mathbf{u}$ is a divergence-free vector field. If there exists $\lambda_0 > 0$ such that $\alpha(L_{\mathbf{u}}) \geq \lambda_0 > 0$, for all sufficiently large R_m, then $\mathbf{u}$ is a **fast kinematic dynamo**.

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The operator $L_{\mathbf{u}} : H^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$ is **not a bounded linear operator** on a Banach space, so the previous theorem does not apply.

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A possible remedy is to consider the operator $L_{\mathbf{u}} : C^\infty(\Omega, \mathbb{R}^3) \to C^\infty(\Omega, \mathbb{R}^3)$ but the theory of semigroups on Fréchet spaces is much more complicated and less developed.

GEARHART–PRÜSS THEOREM: A NEW DEFINITION OF A FAST KINEMATIC DYNAMO

Oxford Mathematical Institute

Trefethen–Embree, Spectra and Pseudospectra, Princeton Univ. Press (2005), Thm. 15.4.

Gearhart–Prüss Theorem: pseudospectra instead of spectra

Let $L : X \to X$ be a closed linear operator on a Hilbert space X that generates a C_0-semigroup, denoted by e^{tL}. Then the following inequality holds:

$$\sup_{t\geq 0} \|e^{tL}\|_{L(X,X)} \geq \alpha_\epsilon(L) := \sup_{\Re(\lambda)\,:\,\lambda\in\sigma_\epsilon(L)} \Re(\lambda),$$

where $\alpha_\varepsilon(L)$ is called the **pseudospectral abscissa** of L and $\sigma_\epsilon(L)$ is the ϵ-pseudospectrum of L, i.e.

$$\sigma_\epsilon(L) := \{\lambda \in \mathbb{C} : \|(\lambda I - L)^{-1}\|_{L(X,X)} > \epsilon^{-1}\}.$$

GEARHART–PRÜSS THEOREM: A NEW DEFINITION OF A FAST KINEMATIC DYNAMO

Mathematical Institute

The operator $L_{\mathbf{u}} : L^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$, defined distributionally as

$$L_{\mathbf{u}}(\mathbf{B}) := \nabla \times (\mathbf{u} \times \mathbf{B}) + R_m^{-1} \nabla \times \nabla \times \mathbf{B},$$

where $\mathbf{u}$ is a divergence-free vector field, is a **closed linear operator** on the Hilbert space $L^2(\Omega, \mathbb{R}^3)$.

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New definition of a fast kinematic dynamo

If there exists $\lambda_0 > 0$ such that $\alpha_\epsilon(L_{\mathbf{u}}) \geq \lambda_0 > 0$, for all sufficiently large R_m and sufficiently small ϵ, then $\mathbf{u}$ is a **fast kinematic dynamo**.

Does our definition coincide with Arnold's?

Does our definition coincide with Arnold's?

Sometimes! They coincide if $\mathbf{u}$ is smooth enough.

DYNAMO EIGENVALUE PROBLEM: REGULARITY

Lemma (regularity of dynamo eigenfunctions)

Let $\Omega \subset \mathbb{R}^3$ be a smooth domain and $\mathbf{u} \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$ be a smooth divergence-free vector field. The space of eigenfunctions $\mathbf{B} \in H_0(\mathrm{curl};\Omega)$ of the dynamo eigenvalue problem is contained in $H^s(\Omega, \mathbb{R}^3)$ for $s \in (1/2, 1]$ and thus the solution operator

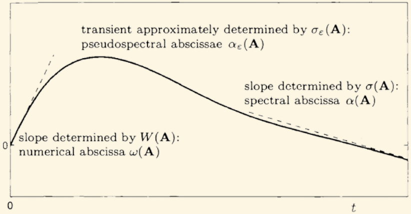
$$T : L^2(\Omega) \to H_0(\mathrm{curl};\Omega) \cap H^s(\Omega;\mathbb{R}^3) \subset L^2(\Omega)$$

associated to the dynamo eigenvalue problem is compact.

For sufficiently smooth velocity fields $\mathbf{u}$ and domains Ω, Arnold's definition of a fast kinematic dynamo gives the same meaningful physical behaviour.

DYNAMO EIGENVALUE PROBLEM: TRANSIENT PHASE AND PSEUDOSPECTRA

In the transient phase (which is better described by pseudospectra), nonlinear effects can become dominant, with the asymptotics described by eigenvalues never reached.



DYNAMO EIGENVALUE PROBLEM: TRANSIENT PHASE AND PSEUDOSPECTRA

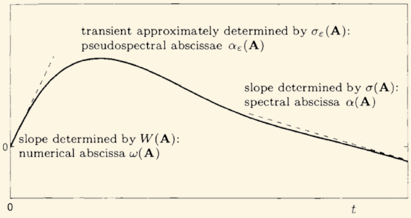
In the transient phase (which is better described by pseudospectra), nonlinear effects can become dominant, with the asymptotics described by eigenvalues never reached.

Theorem (Thm. 15.2 in Trefethen & Embree)

Let $L : X \to X$ be a closed linear operator on a Hilbert space X that generates a C_0-semigroup, denoted by e^{tL}. Then for any $\tau > 0$:

$$\sup_{0\leq t\leq \tau} \|e^{tL}\|_{L(X,X)} \geq \left(e^{\alpha_\epsilon(L)\tau}\right)\left(1 + \frac{e^{\alpha_\epsilon(L)\tau}}{K}\right)^{-1}$$

where $K/\alpha_\epsilon(L) = \|(\lambda I - L)^{-1}\|_{L(X,X)}$ where λ is an eigenvalue such that $\Re(\lambda) = \alpha_\epsilon(L)$.



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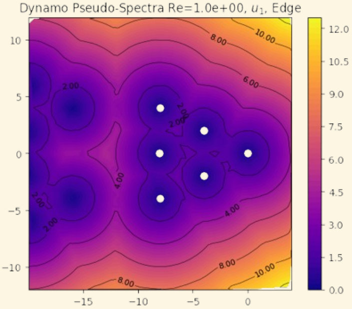

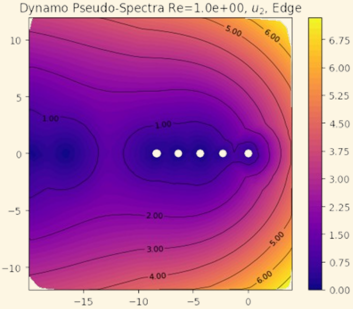

Zeldovich's antidynamo theorem

If the velocity field $\mathbf{u}$ is two-dimensional, then no magnetic field can be sustained by the dynamo mechanism.

UNIFORM CONVERGENCE OF $T_n \to T$

4

Our main result required the uniform convergence of $\mathbf{T_n} \to \mathbf{T}$.

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We prove this with an extension of Kolata's argument.

KOLATA'S ARGUMENT FOR UNIFORM CONVERGENCE

Kolata, Numer. Math. , 29 (1978).

Let $T : H \to X \subset H$ be the solution operator associated to the weak formulation of an eigenvalue problem and let $X_n \subset X \subset H$ be finite-dimensional.

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- Consider the projection operator $\Pi_n : X \to X_n \subset H$ defined via the bilinear form $\circ(\cdot,\cdot)$, i.e.

$$\circ(\Pi_n u, v_n) = \circ(u, v_n) \quad \forall\, v_n \in X_n.$$

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Assume that Π_n converges pointwise to the identity operator $I : H \to H$, i.e.

$$\lim_{n\to\infty} \|\Pi_n u - u\|_H = 0 \quad \forall\, u \in H.$$

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- By the Banach–Steinhaus theorem, the sequence of operators $\{\Pi_n\}_{n\in\mathbb{N}}$ is uniformly bounded, i.e. there exists $C > 0$ such that $\|\Pi_n\|_{\mathcal{L}(X,H)} \leq C, \quad \forall\, n \in \mathbb{N}$.

KOLATA'S ARGUMENT FOR UNIFORM CONVERGENCE

We assume that the discrete solution operator $T_n : H \to H$ is defined as $T_n := \Pi_n T$.

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- We pick a sequence $\{f_n\}_{n\in\mathbb{N}}$ in H such that $\|f_n\|_H = 1$ and $Tf_n \in X_n$ satisfying

$$\|(T - T_n)\|_{\mathcal{L}(H,H)} \leq \|(T - T_n)f_n\|_H + \frac{1}{n}.$$

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- By compactness of T, there exists a subsequence $\{Tf_{n_k}\}_{k\in\mathbb{N}}$ converging to some $w \in X$.

$$\begin{aligned}\|(T - T_{n_k})f_{n_k}\|_H = \|(I - \Pi_{n_k})Tf_{n_k}\|_H &\leq \|(I - \Pi_{n_k})(Tf_{n_k} - w)\|_H + \|(I - \Pi_{n_k})w\|_H \\ &\leq C\|Tf_{n_k} - w\|_X + \|(I - \Pi_{n_k})w\|_H \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

KOLATA'S ARGUMENT FOR UNIFORM CONVERGENCE

Laplace eigenproblem: $\mathbf{X} = \mathbf{L_2(\Omega)}$, $\circ(\cdot,\cdot) = \mathbf{a}(\cdot,\cdot)$, $\mathbf{X_n} = \mathbf{P_1}$

In this case X_n is dense in X as $n \to \infty$ and the projection operator Π_n converges pointwise to the identity operator $I : X \to X$.

The operator $T : X_n \subset L^2(\Omega) \to L^2(\Omega)$ is **compact** since $H^1(\Omega) \subset\subset L^2(\Omega)$.

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The operator $T : X_n \subset L^2(\Omega) \to L^2(\Omega)$ is **compact** since $H^1(\Omega) \subset\subset L^2(\Omega)$.

Maxwell eigenproblem: $\mathbf{X} = \mathbf{L^2}(\mathbf{\Omega})$, $\circ(\cdot,\cdot) = \mathbf{c}(\cdot,\cdot)$, $\mathbf{X_n} = \mathbf{P_1}$

Once again X_n is dense in X as $n \to \infty$ and the projection operator Π_n converges pointwise to the identity operator $I : X \to X$.

However, the operator $T : X_n \subset L_2(\Omega) \to L_2(\Omega)$ is **not compact** since only $H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \subset\subset L_2(\Omega)$.

DISCRETE COMPACTNESS

Kikuchi, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. , 36 (1989).

Definition (Discrete Compactness Property)

A sequence of finite element spaces $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \subset X \subset H$ is said to satisfy the **discrete compactness property** with respect to X and the pivot space H if for any sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ bounded with respect to the norm $\|\cdot\|_X$, there exists a subsequence $\{\mathbf{u}_{n_k}\}_{k\in\mathbb{N}}$ that converges strongly in H.

$\mathbf{X} = \mathbf{L}^2(\boldsymbol{\Omega})$, $\circ(\cdot,\cdot) = c(\cdot,\cdot)$, $\mathbf{X_n} = \mathbf{Ned_1}$

The space $\mathbf{Ned}_1$ satisfies the discrete compactness property with respect to $H_0(\mathrm{curl};\Omega)$ and the pivot space $L^2(\Omega,\mathbb{R}^3)$.

KOLATA ARGUMENT FOR UNIFORM CONVERGENCE

Ern–Guermond, Springer, (2021).

- Assume that we have a projection operator $\Pi_n : H \to X_n$ such that $\|\Pi_n\|_{\mathcal{L}(H,H)} \leq C$ and $T_n \coloneqq \Pi_n T$.

  The construction of such operators is standard in finite element exterior calculus.

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- If we have the **discrete compactness property** for the sequence of finite element spaces $\{X_n\}_{n\in\mathbb{N}}$, then we can extract from $\{Tf_n\}_{n\in\mathbb{N}}$ a subsequence $\{Tf_{n_k}\}_{k\in\mathbb{N}}$ converging strongly in H to some $w \in H$.

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\|(T - T_{n_k})f_{n_k}\|_H = \|(I-\Pi_{n_k})Tf_{n_k}\|_H &\leq \|(I-\Pi_{n_k})(Tf_{n_k} - w)\|_H + \|(I-\Pi_{n_k})w\|_H \\
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The standard available theory for the convergence of FEM approximations of non-selfadjoint compact eigenvalue problems is Osborn's theory.

Osborn, Math. Comp. 29 (1975).

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Osborn's theorem about eigenvalues can be proven as a corollary of our main result about pseudospectra.

OSBORN THEORY FOR NON-SELFADJOINT EIGENVALUE PROBLEMS

Oxford Mathematical Institute

Theorem (Osborn's Theory for Non-Selfadjoint Eigenvalue Problems)

Let $T : H \to H$ be a compact linear operator on a Hilbert space H and consider a sequence of finite-rank operators $T_n : H \to H$ such that

$$\lim_{n\to\infty} \|T_n - T\|_{L(H,H)} = 0.$$

Then the spectrum of the operators T is **convergently approximated** by the spectrum of the operators T_n.

Let μ be a nonzero eigenvalue of T with algebraic multiplicity m and let Γ be a circle centered at μ which lies in $\rho(T)$ and which encloses no other points of $\sigma(T)$. The spectral projection associated with μ and T is defined by

$$E = E(\mu) = \frac{1}{2\pi i}\int_\Gamma R_z(T)\,dz.$$

E is a projection onto the space of generalized eigenvectors associated with μ and T. For n sufficiently large, $\Gamma \subset \rho(T_n)$ and the spectral projection,

$$E_n = E_n(\mu) = \frac{1}{2\pi i}\int_\Gamma R_z(T_n)\,dz,$$

exists; E_n converges to E pointwise and $\{E_n\}$ is collectively compact; and $\dim R(E_n(\mu)) = \dim R(E(\mu)) = m$, where R denotes the range. E_n is the spectral projection associated with T_n and the eigenvalues of T_n which lie in Γ, and is a projection onto the direct sum of the spaces of generalized eigenvectors corresponding to these eigenvalues. Thus, counting according to algebraic multiplicities, there are m eigenvalues of T_n in Γ; we denote these by $\mu_1(n), \ldots, \mu_m(n)$. Furthermore, if Γ' is another circle centered at μ with an arbitrarily small radius, we see that $\mu_1(n), \ldots, \mu_m(n)$ are all inside of Γ' for n sufficiently large, i.e., $\lim_{n\to\infty}\mu_j(n) = \mu$ for $j = 1, \ldots, m$. $R(E)$ and $R(E_n)$ are invariant subspaces for T and T_n, respectively, and $TE = ET$ and $T_nE_n = E_nT_n$. We will also use the fact that $\{R_z(T_n)\colon z \in \Gamma, n \text{ large}\}$ is bounded.

OSBORN THEORY FOR NON-SELFADJOINT EIGENVALUE PROBLEMS

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For non-selfadjoint eigenvalue problems, Osborn's theory also provides the rate of convergence for the arithmetic mean of the discrete eigenvalues.
However, no estimates are available for the individual eigenvalues.

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Open problem: can we provide estimates for the individual eigenvalues of non-selfadjoint eigenvalue problems?

Partial result: In the case that $\mathbf{u} = \nabla\phi$ for some smooth function $\phi : \Omega \to \mathbb{R}$, so that the dynamo eigenvalue problem has only real eigenvalues, we can provide estimates for the rate of convergence of individual eigenvalues.

KIKUCHI FORMULATION

The Kikuchi formulation of the dynamo eigenvalue problem reads is a mixed formulation, so we cannot immediately write $T_n = \Pi_n T$.

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Boffi–Brezzi–Gastaldi theory

A mixed eigenvalue problem of this type has **convergent approximations** if:

- The bilinear form $c(\cdot,\cdot)$ is **coercive** on the kernel of $d(\cdot,\cdot)$, denoted $\mathbf{K}_h$.
- The **weak approximability condition** holds, i.e. for any $(\mathbf{B},\psi)$ that solves the continuous Kikuchi formulation, we have

$$\sup_{\mathbf{B}_h \in \mathbf{V_h}} \frac{d(\mathbf{B}_h, \phi_h)}{\|\mathbf{B}_h\|_{H(\mathrm{curl},\Omega)}} \xrightarrow[h\to 0]{} 0.$$

- The **strong approximability condition on the kernel** holds, i.e. for any $\mathbf{B}$ that solves the continuous Kikuchi formulation, we have

$$\inf_{\mathbf{B}_h \in \mathbf{K_h}} \|\mathbf{B} - \mathbf{B}_h\|_{H(\mathrm{curl},\Omega)} \xrightarrow[h\to 0]{} 0.$$

KIKUCHI FORMULATION

Boffi–Hu–Liang–Z, manuscript in preparation.

Lemma

There exists a constant $\alpha > 0$ independent of h such that

$$c(\mathbf{B}_h, \mathbf{D}_h) + (\mathbf{u} \times \mathbf{B}_h, \nabla \times \mathbf{D}_h)_{L^2(\Omega)} + \alpha m(\mathbf{B}_h, \mathbf{D}_h)$$

is **coercive** on the kernel of $d(\cdot, \cdot)$.

Using the **discrete compactness property** of the $\mathbf{Nec}_1$ element space $\mathbf{Ned}_1$, we can prove the **strong approximability condition** and **weak approximability condition** for the Kikuchi formulation of the **shifted** dynamo eigenvalue problem.

THANK YOU!

Structure-preserving FEEC methods for spectral and pseudospectral analysis of dynamo operators

Umberto Zerbinati*, joint work with: Daniele Boffi†, Kaibo Hu*, Yizhou Liang*, Stefano Zampini†