Time-harmonic nematoacoustics: The nematic Helmholtz–Korteweg equation

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Nematoacoustics in a nutshell

Nematic liquid crystals are fluids made of rod-like molecules that locally align along a unit director field $\boldsymbol{n} \in \mathbb{S}^{d-1}$. They exhibit anisotropic acoustic, elastic, and optical properties, all of which can be *tuned by external electromagnetic fields* [1, 2]. The acoustic anisotropy was first reported by Mullen, Lüthi and Stephen [3]: sound propagates faster along $\boldsymbol{n}$ than across it. Conversely, sufficiently strong ultrasonic waves can themselves *realign* the director, producing a measurable acousto-optic response [4]. This makes nematic-Korteweg fluids natural candidates for the design of **tunable acoustic devices**.

Virga's theory of nematoacoustics

A Korteweg fluid [5] carries a density-gradient stress

$$\underline{\underline{\sigma}}^{(K)} = -pI - u_1\rho\,(\nabla\rho\otimes\nabla\rho), \tag{1}$$

modelling capillarity and phase mixtures. Following Virga [2], the nematic-Korteweg stress augments (1) with an anisotropic contribution along $\boldsymbol{n}$:

$$\underline{\underline{\sigma}}^{(V)} = -pI - u_1\rho\,(\nabla\rho\otimes\nabla\rho) - u_2\,(\nabla\rho\cdot\boldsymbol{n})\,\nabla\rho\otimes\boldsymbol{n}, \tag{2}$$

with material constants $u_1, u_2 > 0$. Classical fluids are recovered in the limit $u_1 = u_2 = 0$.

The nematic Helmholtz–Korteweg equation

Linearising the Euler–Korteweg system around a uniform state and inserting the time-harmonic ansatz $\rho = \rho_0(1+\Re[u(\boldsymbol{x})e^{-i\omega t}])$ yields the *nematic Helmholtz–Korteweg equation*

$$\alpha\,\Delta^2 u + \beta\,\nabla\cdot\nabla\big(\boldsymbol{n}^\top(\mathcal{H}u)\,\boldsymbol{n}\big) - \Delta u - k^2 u = f \quad \text{in } \Omega, \tag{3}$$

where $\mathcal{H}u$ is the Hessian of u, k is the wave number, and $\alpha \gg \beta \geq 0$ are the acoustic susceptibilities [6]. Being fourth order, (3) needs *two* boundary conditions on $\partial\Omega$, mirroring the classical Helmholtz pairs:

- **sound-soft:** $u = 0, \quad \alpha\Delta u + \beta\,\boldsymbol{n}^\top(\mathcal{H}u)\boldsymbol{n} = 0;$
- **sound-hard:** $\nabla u\cdot\boldsymbol{\nu} = 0, \quad \nabla\big(\alpha\Delta u + \beta\,\boldsymbol{n}^\top(\mathcal{H}u)\boldsymbol{n}\big)\cdot\boldsymbol{\nu} = 0;$
- **impedance:** $\nabla u\cdot\boldsymbol{\nu} = i\theta\,u,$ $\nabla\big(\alpha\Delta u + \beta\,\boldsymbol{n}^\top(\mathcal{H}u)\boldsymbol{n}\big)\cdot\boldsymbol{\nu} = i\theta\big(\alpha\Delta u + \beta\,\boldsymbol{n}^\top(\mathcal{H}u)\boldsymbol{n}\big).$

Anisotropic dispersion

A plane wave $u(\boldsymbol{x}) = e^{i\boldsymbol{d}\cdot\boldsymbol{x}}$ with $|\boldsymbol{d}| = d$ satisfies the dispersion relation

$$\alpha\,d^4 + \beta\,d^2(\boldsymbol{d}\cdot\boldsymbol{n})^2 + d^2 - k^2 = 0. \tag{4}$$

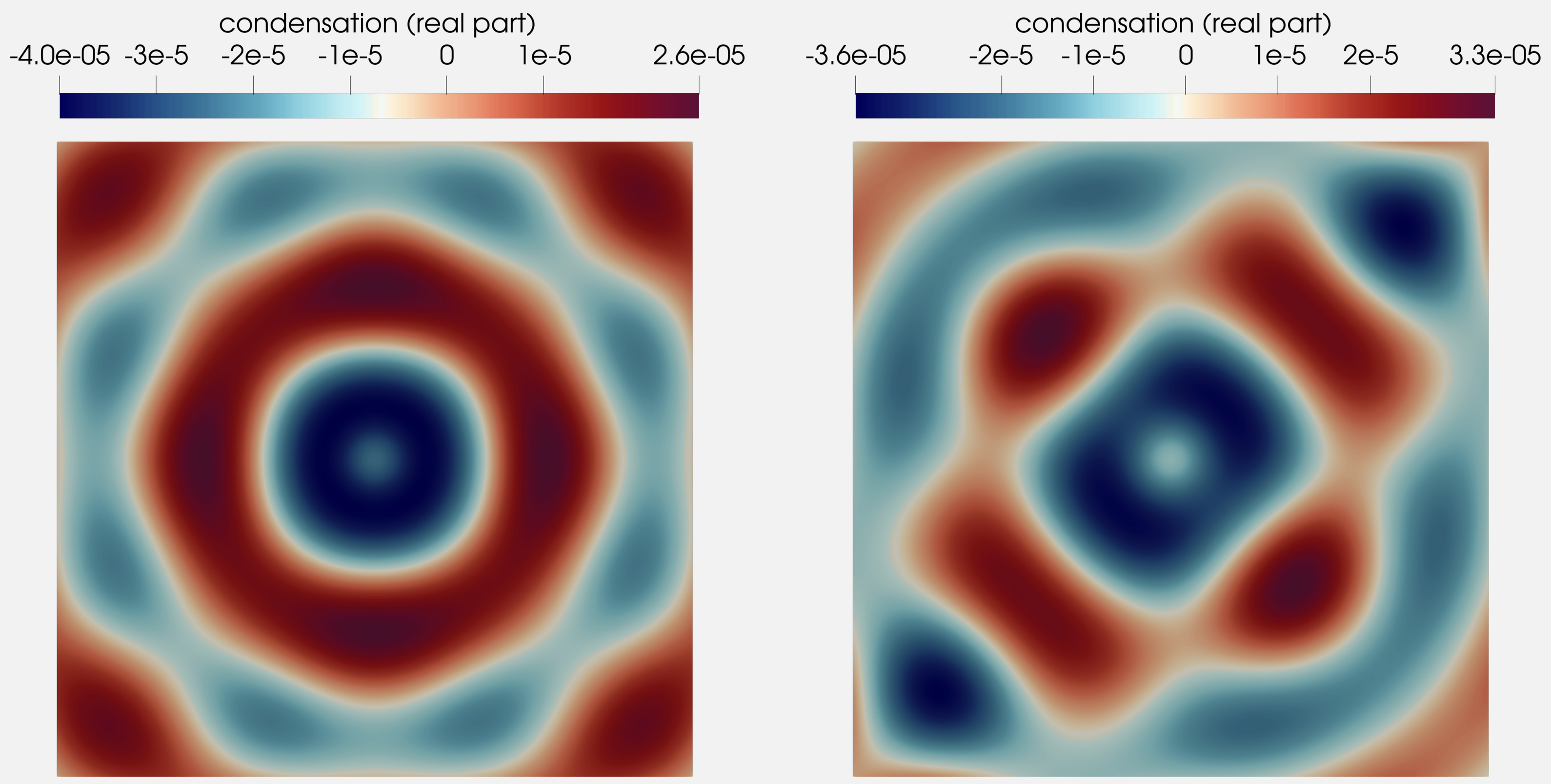
With ξ the angle between $\boldsymbol{d}$ and $\boldsymbol{n}$, the speed of sound depends on $\cos^2\xi$: **fastest parallel to $\boldsymbol{n}$, slowest orthogonal** – in agreement with Virga's derivation [2] and the Mullen–Lüthi–Stephen measurement.

H^2-conforming discretisation

Equation (3) is fourth order: we discretise it with H^2-conforming finite elements (Argyris, $p \geq 5$, rate h^{p-1} in H^2 or Hsieh–Clough–Tocher macro, $p \geq 3$, rate h^{p-2}). Boundary conditions are imposed weakly via Nitsche's method. Well-posedness follows from a T-*coercivity* argument. Full analysis can be found in the companion paper [7]. All the numerical experiments presented here were implemented using Firedrake.

Anisotropic speed of sound

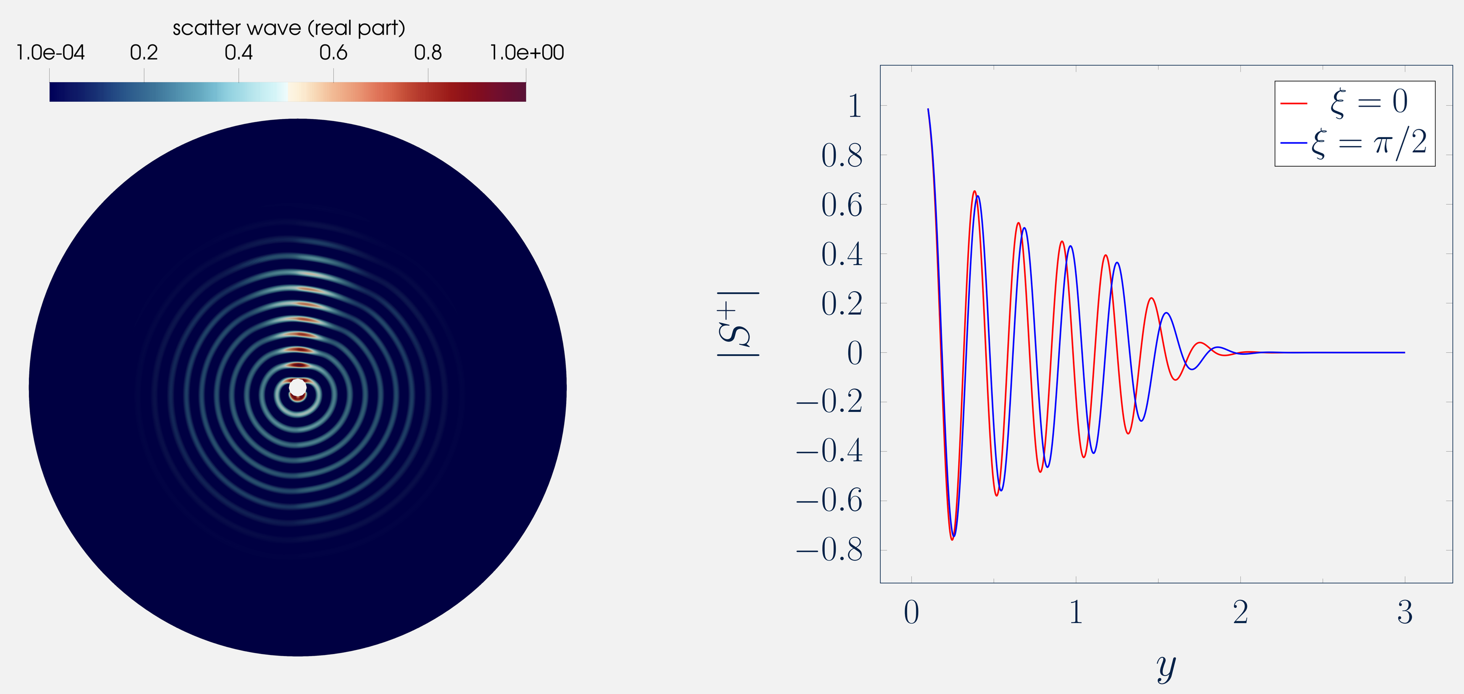
A symmetric Gaussian source confirms the dispersion anisotropy on localised pulses: the wavefront **stretches along $\boldsymbol{n}$**.



Left: isotropic baseline ($\beta = 0$). Right: $\boldsymbol{n} = (1,1)/\sqrt{2}$. $k = 40$, $\alpha = 10^{-2}$, $\beta = 5\cdot 10^{-3}$.

Scattering by a circular obstacle

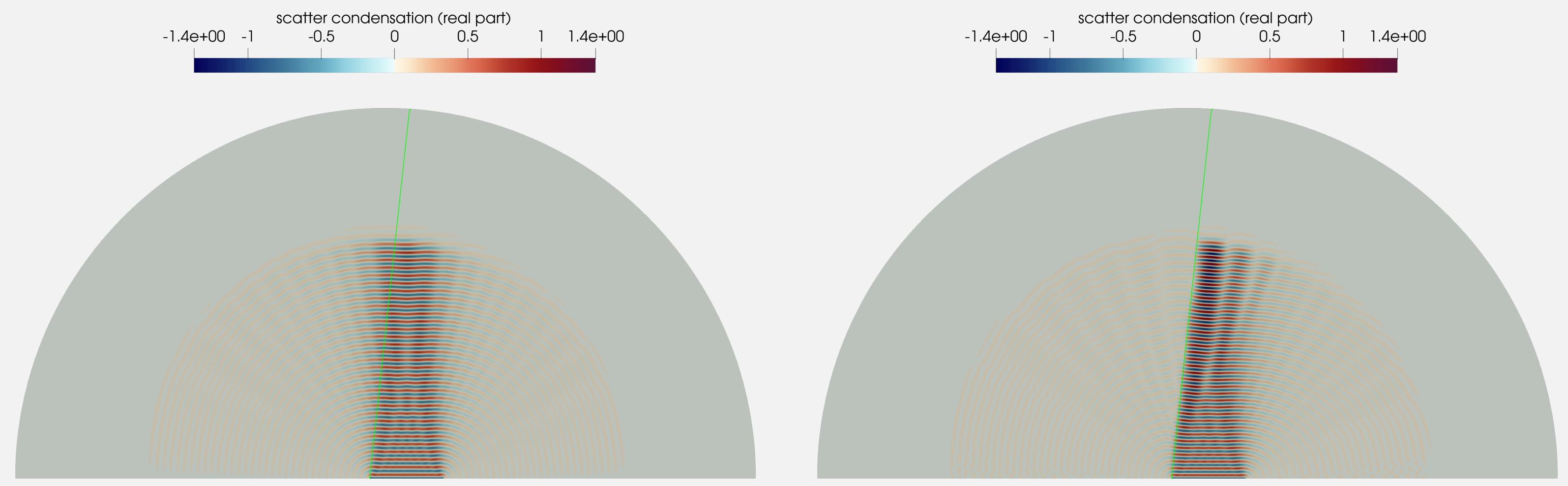
Plane wave scattering off a sound-soft disc. The nematic effect persists **even for sound-soft boundaries**: the scattered amplitude is larger when the wave is orthogonal to $\boldsymbol{n}$.



Scattered field (left); amplitude on the y-axis is larger when $\boldsymbol{d}\perp\boldsymbol{n}$ (right).

Reflection from a director jump

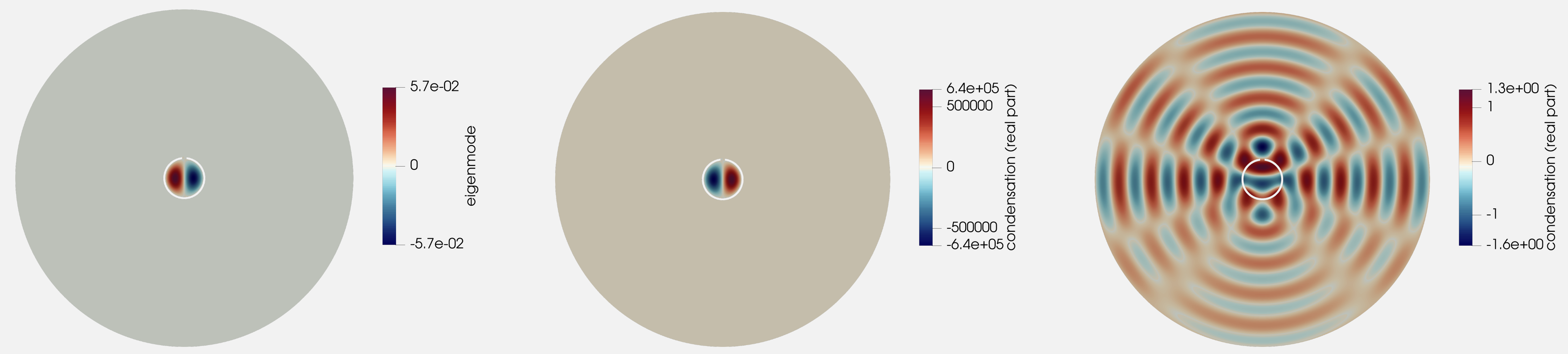
Considering the Gaussian beam and sound-soft boundary we change $\boldsymbol{n}$ only. *Smooth* director (left): the wave propagates undisturbed. *Director jump* across the green line – $\boldsymbol{n}\parallel\boldsymbol{d}$ on the left and $\boldsymbol{n}\perp\boldsymbol{d}$ on the right (right panel): the wave is **partially reflected** along the line, despite the sound-soft boundary. The effect vanishes for the classical Helmholtz equation.



Smooth director (left) vs. director jump (right).

A tunable acoustic resonator

Cavity eigenvalue $\lambda_h \approx 21.83 - 2.43\cdot 10^{-9}i$ for $\boldsymbol{n} = (1,0)$, excited at $k = \sqrt{\Re(\lambda_h)+\varepsilon}$: $\sim 10^7\times$ **amplification**. Rotating to $\boldsymbol{n} = (0,1)$ takes the spectrum off-resonance.



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