Some remarks on normality and non-normality

Umberto Zerbinati*

** Mathematical Institute – University of Oxford, United Kingdom*

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UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

A VISIT TO EDINBURGH

All the content of this slides comes from a short-course K. Hu invited me to give at the University of Edinburgh. You can find the lectures notes and all proofs at:

`https://www.uzerbinati.eu/teaching/spectral_theory/`

THE ADJOINT: A BRIEF REMINDER

Let $(X, \langle \cdot, \cdot \rangle)$ and $(Y, \langle \cdot, \cdot \rangle)$ be two Hilbert spaces and $A : X \to Y$ be a bounded linear operator.

Adjoint Operator

The adjoint operator $A^* : Y \to X$ is defined by the relation
$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \quad \forall x \in X, y \in Y.$$

- Commonly the adjoint operator is computed integrating by parts, but this is only the **formal adjoint**.
- There are different definitions of adjoint operators depending if we are working in **Banach** or **Hilbert** spaces. The two notions are reconciled via the use of the Riesz representation theorem.
- Most differential operators are not bounded operators, they rather belong to the class of **unbounded densely defined operators**. For this reason we will mostly work with the **solution operator** associated to the differential operator.

NORMALITY: INTEGRATION BY PARTS DOESN'T TELL THE WHOLE STORY

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Normal Operator

An operator $A : X \to Y$ is said to be normal if $A^*A = AA^*$.

Let us consider the following example:

$$T : L^2(\Omega) \to L^2(\Omega), \quad f \mapsto u \text{ such that } \mathcal{L}u := -\Delta u + \underline{w} \cdot \nabla u = f,\ u|_{\partial\Omega} = 0.$$

If $\underline{w} \in \mathbb{R}^3$ is a constant vector, we can compute the formal adjoint of the PDE, i.e.

$$\langle \mathcal{L}u, v\rangle = \int_\Omega (-\Delta u + \underline{w} \cdot \nabla u)v = \int_\Omega u(-\Delta v + \underline{w} \cdot \nabla v) = \langle u, \mathcal{L}^* v\rangle.$$

Even if $\mathcal{L}\mathcal{L}^* = \mathcal{L}^*\mathcal{L}$, the solution operator T is not normal, neither is the unbounded operator: $u \mapsto -\Delta u + \underline{w} \cdot \nabla u$.

CONSEQUENCE OF NON-NORMALITY FROM AN NLA PERSPECTIVE

▷ Lazzarino, L., Nakatsukasa, Y. and $\sim$, 2025. Preconditioned normal equations for solving discretised partial differential equations. arXiv preprint arXiv:2502.17626.

Non-normality

Let us consider a finite rank approximation of the operator T, here on denoted by T_n. Since the operator T is not normal, we should expect that also the finite rank approximation T_n is not normal. This implies if T_n is diagonalizable, i.e. $T_n = Q_n \Lambda_n Q_n^{-1}$, then the condition number of Q_n can be arbitrarily large.

- The fact that the condition of number of Q_n can get arbitrarily large implies the spectrum of T_n, i.e. Λ_n, does't give us any bound on the convergence rate of a Krylov solver.
- A possible cure for this is to consider the normal equations associated with the PDE.
- We need an efficient way to precondition the normal equations.

▷ Lazzarino, L., Nakatsukasa, Y. and ∼ , 2025. Preconditioned normal equations for solving discretised partial differential equations. arXiv preprint arXiv:2502.17626.

Primal Dual Error in the Normal Equation

There is a primal dual error in the classical formulation of the normal equations.

$$V_h \subset H^1_0(\Omega) \xrightarrow{A} H^{-1} \subset V_h' \qquad V_h \subset H^1_0(\Omega) \xrightarrow{A^T} H^{-1} \subset V_h'$$

To make sense of the normal equations we need to consider a Riesz map $T : V_h' \to V_h$.

$$V_h \subset H^1_0(\Omega) \xrightarrow{A} H^{-1} \subset V_h' \xrightarrow{\quad T \quad} V_h \subset H^1_0(\Omega) \xrightarrow{A^T} H^{-1} \subset V_h'$$

$$\underline{\underline{A}}^T T \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^T T \underline{b},$$

SPECTRUM AND SPECTRAL THEOREM

▷ Trefethen, L.N. and Embree, M., 2020. Spectra and pseudospectra: the behavior of nonnormal matrices and operators.

It is well known that if $T : X \to X$ is a normal operator its behaviour can be characterized in great detail by its spectrum. First introduce the resolve, i.e. $R_T(z) := (T - z)^{-1} : X \to X$.

Spectrum

We call the resolvent set of T, here denoted ρ_T the set of $z \in \mathbb{C}$ such that $(T - z) : X \to X$ is a bijection, and thus the resolvent operator $R_T(z)$ is well-defined. The spectrum of T is the complement of the resolvent set, here denoted σ_T.

- Given a normal compact operator T and assuming X is separable, there exists a Hilbert basis for X formed by the eigenvectors of T.

A perhaps more interesting tool in the context of non-normal operator is the pseudospectra.

Pseudospectrum

We call the ε-pseudospectrum of T, denoted $\sigma_{T,\varepsilon}$, the set of $z \in \mathbb{C}$ such that $\|(T - z)^{-1}\| \geq \frac{1}{\epsilon}$ for some $\epsilon > 0$.

The pseudo spectra is giving us an upper-bound on the continuity constant of the resolvent set. Fixed a tolerance ε for all $z \in \sigma_{T,\varepsilon}$ the generalised Helmholtz problem $(T - z)u = f$ has a continuous dependence on the data of the form, $\|u\|_X \leq \varepsilon^{-1}\|f\|_X$.

- If the operator T is not normal, we are no longer guaranteed the existence of a spectral Hilbert basis.
- The pseudo-spectra contains all balls of radius ε around the eigenvalues of T, but it can also be much larger.

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▷ Osborn, J.E., 1975. Spectral approximation for compact operators. Mathematics of computation, 29(131), pp.712-725.
▷ Boffi, D., Brezzi, F. and Gastaldi, L., 2000. On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form. Mathematics of computation, 69(229), pp.121-140.

Osborn developed a theory to describe the approximation of the spectra of a finite element discretization $\{T_n\}_{n\in\mathbb{N}}$. Osborn theory requires that the T_n converges uniformly to T.

Lower approximability of the pseudo-spectra (work in progress with L. Lazzarino, K. Hu and P.E. Farrell)

Fixed any $\varepsilon > 0$ and $\delta \in (0,\varepsilon)$ it exists an ℓ such that for any $n > \ell$ the following inclusion holds: $\sigma_{T_n,\delta} \subset \sigma_{T,\varepsilon}$.

APPROXIMATION OF THE PSEUDOSPECTRA

- This tool can be used to prove the absence of spurious modes for **uniformly convergent** discretisations, and is under the hood in Osborn theory.
- Under the hypothesis that $T_n = T\Pi_n$ with Π_n being a bounded projection operator such that Π_n converges point-wise to the identity, we can obtain more quantitative estimates.

An interesting consequence of the above result is that the approximation of the pseudo-spectra gives some information also about the convergence of the Helmholtz problem.

Generalised Helmholtz problem (work in progress with L. Lazzarino, K. Hu and P.E. Farrell)

Under the hypothesis that T_n converges uniformly to T we can use Osborn identity to obtain for any $z \in \sigma_{T,\varepsilon}$ and sufficiently large n that

$$\|R_T(z) - R_{T_n}(z)\|_{\mathcal{B}(X,X)} \leq \varepsilon \|T - T_n\|_{\mathcal{B}(X,X)},$$

- Similar to the **T-coercivity** we can prove the well-posedness of the associated Helmholtz problem, only discussing the spectrum of T even in the non-normal context.

FROM THE DISCRETE TO THE CONTINUOUS: AN EXAMPLE

> Drazin, M.P. and Haynsworth, E.V., 1962. Criteria for the reality of matrix eigenvalues. Mathematische Zeitschrift, 78(1), pp.449-452.

- Let us consider once again the advection diffusion equation, one can prove via a scaling argument that for FEM discretisation the imaginary part of the eigenvalues vanishes as $n \to \infty$.
- Thus we know that $\sigma_T \subset \mathbb{R}$ and therefore it exists a self-adjoint operator S such that $TS = ST*$.
- This implies the existence of transformation of that reduces the PDE to a self-adjoint one, e.g. $v(x) = e^{-ax}u(x)$.
- The existence of uniformly convergent discretisation for the advection diffusion problem, with positivity preserving basis, such that T_n is an M-matrix implies that the semigroup generated by T is positive.

THANK YOU!

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