Embedded Trefftz Discretisations for the Maxwell Eigenvalue Problem with Applications to MHD

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The idea behind DG-Trefftz methods is to consider a discontinuous Galerkin method where the local approximation spaces are made of functions that are piecewise solutions of the target PDE. For example, let us consider the Laplace equation,

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where $\mathbb{P}^p(K)$ is the space of polynomials of degree at most p on the element K. The global discrete space is then defined as

$$\mathbb{T}_h = \{v_h \in L^2(\Omega) : v_h|_K \in \mathbb{T}^p(K), \forall K \in \mathcal{T}_h\}.$$

No conformity is imposed across element interfaces in the space $\mathbb{T}_h$, hence a DG formulation is needed to enforce the continuity of the solution. We thus consider the following DG formulation: find $u_h \in \mathbb{T}_h$ such that

$$\int_{\mathcal{T}_h} \nabla u_h \cdot \nabla v_h \, dx - \int_{\mathcal{F}_h} (\llbracket u_h \rrbracket \cdot \{\nabla v_h\} + \llbracket v_h \rrbracket \cdot \{\nabla u_h\}) \, ds + \int_{\mathcal{F}_h} \frac{\sigma p^2}{h} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket \, ds = - \int_{\partial\Omega} g(\partial_n v_h) \, ds,$$

$$- \int_{\partial\mathcal{T}_h} (u_h \partial_n v_h + v_h \partial_n u_h) \, ds + \int_{\partial\mathcal{T}_h} \frac{\sigma p^2}{h} u_h v_h \, ds,$$

for all $v_h \in \mathbb{T}_h$, where $\mathcal{F}_h$ is the set of all faces in the mesh $\mathcal{T}_h$, σ is a positive penalty parameter, and h is the mesh size

AN EIGENVALUE PROBLEM

Since we have assembled the stiffness matrix, we can also assemble the mass matrix and consider the following eigenvalue problem: find $(\lambda_h, u_h) \in \mathbb{R} \times \mathbb{T}_h$ such that

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$$- \int_{\partial \mathcal{T}_h} (u_h \partial_n v_h + v_h \partial_n u_h) \, ds + \int_{\partial \mathcal{T}_h} \frac{\sigma p^2}{h} u_h v_h \, ds = \lambda_h \int_{\mathcal{T}_h} u_h v_h \, dx,$$

for all $v_h \in \mathbb{T}_h$. The mass matrix is the standard DG mass matrix, i.e. $\int_{\mathcal{T}_h} u_h v_h \, dx$.

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- The mass matrix $\underline{\underline{M}}$ only need to be the DG mass matrix, since conformity is already imposed in the stiffness matrix.
- The stiffness matrix is parameter dependent, i.e. $\underline{\underline{K}} = \underline{\underline{K}}_1 + \sigma \underline{\underline{K}}_2$.

PARAMETER DEPENDENCE

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Exact	2	5	5	8	10	10	13	13	17	17
$\sigma = 0.3\,(!)$	2.00	3.81	5.01	5.01	6.12	8.03	9.41	10.04	10.05	11.28
$\sigma = 1.0$	2.00	5.01	5.01	8.03	10.04	10.05	13.08	13.08	17.14	17.14

- The parameter dependence is very benign !

EMBEDDED TREFFTZ METHOD: GLOBAL

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- We here consider the "ambient space" V_h made of standard DG polynomials of degree p on the mesh $\mathcal{T}_h$, with basis $\{\phi_j\}_{j=1}^{N_{dof}}$. There is a canonical isomorphism between $\mathbb{R}^{N_{dof}}$ and V_h, i.e.

$$\mathcal{G} : \mathbb{R}^{N_{dof}} \to V_h, \quad \mathbf{c} \mapsto \sum_{j=1}^{N_{dof}} c_j \phi_j.$$

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- Given an operator $\mathcal{L}$, we construct the matrix,

$$\underline{\underline{W}}_{ij} = \int_{\mathcal{T}_h} \langle \mathcal{L}\phi_j, \mathcal{L}\phi_i \rangle \, dx, \quad 1 \leq i, j \leq N_{dof},$$

notice that in V_h the operator $\mathcal{L}$ has kernel $\mathcal{G}(\ker(\underline{\underline{W}}))$. We are interested in an orthogonal projector onto $\ker(\underline{\underline{W}})$, which can be computed via the SVD of $\underline{\underline{W}}$.

EMBEDDED TREFFTZ METHOD: ELEMENT-WISE

- Notice that since we don't have global conformity, we can proceed element wise, i.e. we can define $\underline{\underline{W}}^{(K)}$ for each element $K \in \mathcal{T}_h$.

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$$\underline{\underline{W}}^{(K)} = \begin{bmatrix} \underline{\underline{U}}_1 & \underline{\underline{U}}_2 \end{bmatrix} \begin{bmatrix} \underline{\underline{\Sigma}}_1 & \\ & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\underline{V}}_1^T \\ \underline{\underline{V}}_2^T \end{bmatrix}, \quad \underline{\underline{T}}^{(K)} = \underline{V}_2$$

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- The element-wise nature of these procedure makes it computationally feasible (and **highly parallelisable**).
- Assembling the stiffness and mass matrices over the local Trefftz spaces formed by $\mathcal{G}(\ker(\underline{\underline{W}}^{(K)}))$ is then straightforward and is equivalent to the eigenvalue problem

$$\underline{\underline{T}}^T \underline{\underline{K}}\, \underline{\underline{T}}\mathbf{u} = \lambda \underline{\underline{T}}^T \underline{\underline{M}}\, \underline{\underline{T}}\mathbf{u}.$$

- λ are the Ritz values of the matrix pencil $(\underline{\underline{K}}, \underline{\underline{M}})$ via the projection $\underline{\underline{T}}$.

SOME NUMERICAL LINEAR ALGEBRA REMARKS

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Poincaré Separation Theorem

Let $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{N_{dof}}$ be the eigenvalues of the symmetric positive definiteness pencil $(\underline{\underline{K}}, \underline{\underline{M}})$ then the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{N_{trf}}$ the Ritz values of the pencil $(\underline{\underline{K}}, \underline{\underline{M}})$ via projection $\underline{\underline{T}}$ satisfy

$$\mu_i \leq \lambda_i \leq \mu_{N_{dofs}-N_{trf}+i}, \quad i = 1, \ldots, m.$$

SAAD'S STYLE ESTIMATE

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Saad's style estimate

Let $P := \underline{\underline{T}}(\underline{\underline{T}}^T \underline{\underline{M}}\,\underline{\underline{T}})^{-1}\underline{\underline{T}}^T$ be the orthogonal projector onto the space spanned the columns of V_0, for every $1 \leq i \leq N_{dofs}$, there exists a constant $1 \leq j \leq N_{trf}$ such that

$$|\mu_i - \lambda_j| \leq (\mu_{N_{dofs}} - \mu_1) \min_{p \in \Pi^{N_{trf}}} \max_{1 \leq k \neq i \leq N_{dofs}} |p(\mu_k)| \frac{\|(\underline{\underline{I}} - \underline{\underline{P}})\mathbf{v}_i\|_{\underline{\underline{M}}}}{\|\underline{\underline{P}}\mathbf{v}_i\|_{\underline{\underline{M}}}}$$

where $\Pi^{N_{trf}}$ is the set of polynomials of degree at most N_{trf} such that $p(\mu_i) = 1$, where $\mathbf{v}$ is the eigenvector associated with the eigenvalue μ_i, i.e. $\underline{\underline{K}}\mathbf{v}_i = \mu_i \underline{\underline{M}}\mathbf{v}_i$.

SAAD'S STYLE ESTIMATE

- The quality of the approximation depends the angle φ such that

$$\tan(\varphi) := \|(I-P)\mathbf{v}\|_{\underline{\underline{M}}} / \|P\mathbf{v}\|_{\underline{\underline{M}}},$$

i.e. if an eigenvector $\underline{\mathbf{v}}$ is well approximated in the space spanned by the columns of V_0, then φ is small and the approximation is good.

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- Notice that since $\underline{\underline{M}}$ is the discrete L^2 inner product, then

$$\|(I-P)\mathbf{v}_i\|_{\underline{\underline{M}}}/\|P\mathbf{v}_i\|_{\underline{\underline{M}}} = \|v_i - w_i\|_{L^2(\Omega)}/\|w_i\|_{L^2(\Omega)},$$

where w_i is the embedding in V_h of the best approximation in the Trefftz space of the eigenfunction associated with μ_i, with respect to the L^2 norm.

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  where w_i is the embedding in V_h of the best approximation in the Trefftz space of the eigenfunction associated with μ_i, with respect to the L^2 norm.
- By means of Taylor expansion arguments, one can show that as $h \to 0$ also $\|v_i - w_i\|_{L^2(\Omega)} \to 0$ and thus the Ritz values converge to the DG eigenvalues.

MAXWELL EIGENVALUE PROBLEM

We consider the Maxwell eigenvalue problem: find $(\omega, \mathbf{u}) \in \mathbb{R} \times \mathbf{H}_0(\mathrm{curl};\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega)$ such that

$$(\mathrm{curl}\ \mathbf{u}, \mathrm{curl}\ \mathbf{v}) = \omega^2(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl};\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega),$$

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- The space $\mathbf{H}_0(\mathrm{curl};\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega)$ is hard to discretise, we are not aware of any conforming finite element space for this space.
- The eigenvalue problem has to be treated with care due to the large kernel of the curl operator, i.e. $\ker(\mathrm{curl}) = \nabla H_0^1(\Omega)$. We have no zero eigenvalues, due the divergence free constraint.

KIKUCHI FORMULATION OF MAXWELL EIGENVALUE PROBLEM

Instead of working directly with the Maxwell eigenvalue problem, we resort to a mixed formulation first proposed by Kikuchi: find $(\lambda, \mathbf{u}, p) \in \mathbb{R} \times \mathbf{H}_0(\mathrm{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$(\mathrm{curl}\ \mathbf{u}, \mathrm{curl}\ \mathbf{v}) + (\mathbf{v}, \nabla p) = \lambda(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl}; \Omega),$$

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- The Lagrange multiplier p enforces the divergence free constraint on $\mathbf{u}$.
- A conforming discretisation of this problem can be obtained by considering **Nédélec** elements for $\mathbf{u}$ and standard **Lagrangian** elements for p.
- Less restrictive conditions have to be imposed on the spaces to ensure absence of spurious modes, with respect to other formulations. In particular, weak and strong approximability conditions ensure the absence of spurious modes (the easyest way to ensure these is via the **inf-sup** condition).

A DISCONTINUOUS GALERKIN KIKUCHI FORMULATION

Mimicking the Kikuchi formulation, we consider the following DG formulation: find $(\lambda_h, \mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$
\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= \lambda_h m_h(\mathbf{u}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
b_h(\mathbf{u}_h, q_h) + c_h(p_h, q_h) &= 0, \quad \forall q_h \in W_h,
\end{aligned}
$$

$$
\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) = &\int_{\mathcal{T}_h} \mathrm{curl}\ \mathbf{u}_h \cdot \mathrm{curl}\ \mathbf{v}_h \, dx - \int_{\mathcal{F}_h} \left([\![\mathbf{u}_h \times \mathbf{n}]\!] \cdot \{\mathrm{curl}\ \mathbf{v}_h\} + [\![\mathbf{v}_h \times \mathbf{n}]\!] \cdot \{\mathrm{curl}\ \mathbf{u}_h\} \right) ds \\
&+ \int_{\mathcal{F}_h} \frac{\sigma p^2}{h} [\![\mathbf{u}_h \times \mathbf{n}]\!] \cdot [\![\mathbf{v}_h \times \mathbf{n}]\!] \, ds - \int_{\partial\mathcal{T}_h} (\mathbf{u}_h \times \mathbf{n}) \cdot (\mathrm{curl}\ \mathbf{v}_h) \, ds \\
&- \int_{\partial\mathcal{T}_h} (\mathbf{v}_h \times \mathbf{n}) \cdot (\mathrm{curl}\ \mathbf{u}_h) \, ds + \int_{\partial\mathcal{T}_h} \frac{\sigma p^2}{h} (\mathbf{u}_h \times \mathbf{n}) \cdot (\mathbf{v}_h \times \mathbf{n}) \, ds, \\
b_h(\mathbf{u}_h, q_h) = &-\int_{\mathcal{T}_h} \mathrm{div}\ \mathbf{u}_h q_h \, dx + \int_{\mathcal{F}_h} [\![\mathbf{u}_h \cdot \mathbf{n}]\!] \{q_h\} \, ds
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$$c_h(p_h, q_h) = \int_{\partial \mathcal{T}_h} \frac{\sigma p^2}{h^2} p_h q_h \, ds, \qquad m_h(\mathbf{u}_h, \mathbf{v}_h) = \int_{\mathcal{T}_h} \mathbf{u}_h \cdot \mathbf{v}_h \, dx.$$

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- We are enforcing the Dirichlet boundary condition on the pressure via an Aubin–Babuska type penalty term.
- Our formulation is slightly different from the one proposed by **Houston–Perugia–Schötzau** where instead the bilinear form $b_h(\cdot,\cdot)$ had the form

$$b_h(\mathbf{u}_h, q_h) = -\int_{\mathcal{T}_h} \mathbf{u}_h \cdot \nabla q_h \, dx + \int_{\mathcal{F}_h} \{\mathbf{u}_h \cdot \mathbf{n}\} [\![q_h]\!] \, ds + \int_{\mathcal{F}_h} \frac{\sigma p^2}{h} [\![p_h]\!] [\![q_h]\!] \, ds.$$

In fact, we have integrated by parts the term $(\mathbf{u}, \nabla q)$ and dropped the interior penalty term.

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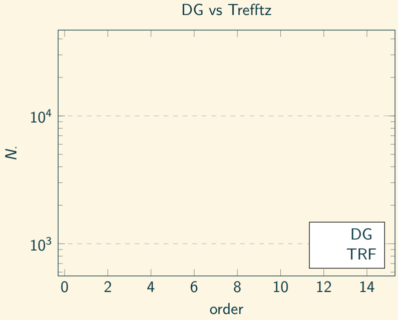
- We consider the following local operator

$$\langle \mathcal{L}\mathbf{u}, \mathbf{v} \rangle = \int_K \text{curl curl } \mathbf{u} \cdot \mathbf{v} \, dx, \qquad \forall \mathbf{v} \in \mathbb{P}^{p-2}(K).$$



At $p = 14$ we have $N_{dofs} = 32400$ for DG and $N_{trf} = 23040$ for Trefftz.

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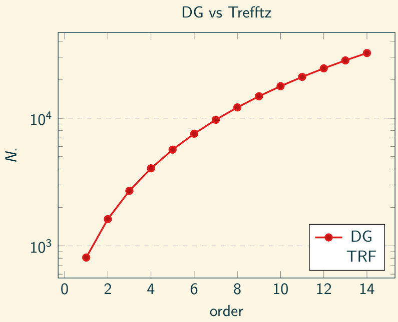
- We then co nstruct the local Trefftz space via the **embedded Trefftz** procedure, i.e. we compute the matrix $\underline{\underline{W}}^{(K)}$ and the space $\mathbb{T}^p(K) = \mathcal{G}(\ker(\underline{\underline{W}}^{(K)}))$.



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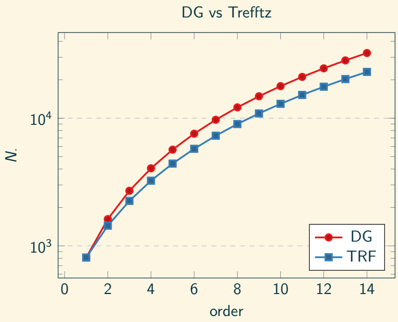
- equivalently, we solve for the eigenvalue problem

$$\underline{\underline{T}}^T \underline{\underline{K}}\, \underline{\underline{T}}\mathbf{u} = \lambda \underline{\underline{T}}^T \underline{\underline{M}}\, \underline{\underline{T}}\mathbf{u},$$

where $\underline{\underline{K}}$ and $\underline{\underline{M}}$ are the stiffness and mass matrices of the DG Kikuchi formulation.



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CONSTRAINED TREFFTZ DISCRETISATION OF KIKUCHI FORMULATION

- The curl curl operator has a large kernel, thus the local Trefftz space $\mathbb{T}^p(K)$ will contain many basis functions, yielding a reduction in the number of degrees of freedom far from optimal.

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- The curl curl operator has a large kernel, thus the local Trefftz space $\mathbb{T}^p(K)$ will contain many basis functions, yielding a reduction in the number of degrees of freedom far from optimal.
- To compensate for this result we impose partial conformity across element interfaces strongly, and thus consider the following Constrained Trefftz space

$$\mathbb{T}^p_c(K) = \{\phi \in V_h : \langle \mathcal{L}\phi_j, \xi \rangle = 0 \; \forall \xi \in \mathbb{Q}_h \text{ and } \exists \varphi \in \mathbb{Z}_h \; : \; c_K(\phi, \psi) = d_K(\varphi, \psi) \forall \psi \in \mathbb{Z}_h\},$$

where $\mathbb{Q}_h$ and $\mathbb{Z}_h$ are respectively **local spaces** used to impose the Trefftz constraint and the conformity constraint, while $c_K(\cdot, \cdot)$ and $d_K(\cdot, \cdot)$ are the local bilinear forms used to impose the conformity constraint.

CONSTRAINED TREFFTZ DISCRETISATION OF KIKUCHI FORMULATION

- To construct the Constrained Trefftz space, on each element we consider the following local linear system

$$\begin{pmatrix} \boxed{\underline{\underline{W}}_K} \\ \\ \boxed{\underline{\underline{C}}_K} \end{pmatrix} \cdot \left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ & \dots u_C \dots & & & \dots u_T \dots & \\ | & | & | & | & | & | \end{array} \right) = \begin{pmatrix} 0 & 0 \\ \boxed{\underline{\underline{D}}_K} & 0 \end{pmatrix}$$

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- Our new projector matrix has now form,

$$\underline{\underline{T}}^{(K)} = \left[\underline{\underline{U}}_C \quad \underline{\underline{U}}_T\right],$$

where $\underline{U}_C$ are the basis functions that satisfy the Trefftz constraint and the trace constraint, while $\underline{U}_T$ are the basis functions that only satisfy the Trefftz constraint and have vanishing trace constraint. Notice that the all linear system can be solve via SVD.

CONSTRAINED TREFFTZ DISCRETISATION OF KIKUCHI FORMULATION

Mathematical Institute

- We begin considering the following constraint operator for $\phi, \varphi \in V_h(K), \mathbf{Z}_h(\partial K) := \mathbb{P}^k(\partial K)$,

$$c_K(\phi, \psi) = \int_{\partial K} (\phi \times n) \cdot (\psi \times n) \, ds, \quad \forall \psi \in \mathbb{Z}_h(K).$$

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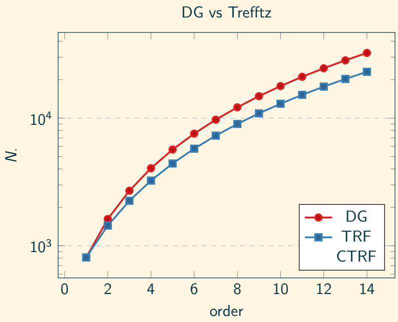

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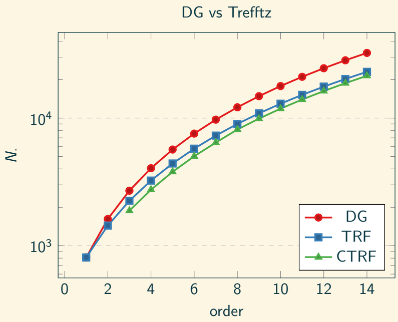

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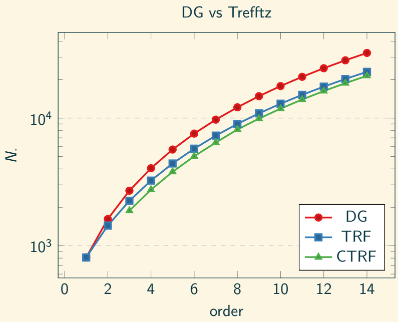

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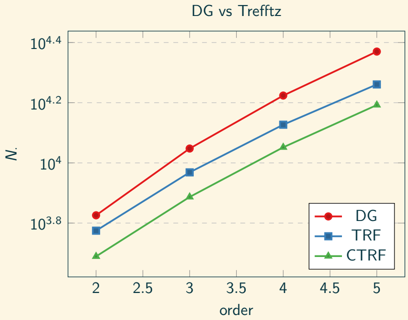

At $p = 5$ we have $N_{dofs} = 23436$ for DG, $N_{trf} = 18228$ for Trefftz and $N_{ctrf} = 15568$ for Constrained Trefftz.

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- If we try to impose partial normal continuity of $\mathbf{u}_h$ across element interfaces together with the tangential continuity, we observe the appearance of spurious modes.

Exact		1	1		2		4	4	5	5	8
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- The appearance of the spurious modes is an example of the fact that Saad's style estimate doesn't guarantee convergence of the Ritz values without spurious modes.

AN MHD EIGENVALUE PROBLEM

Following the work by Arnold we are interested in the following eigenvalue problem:

$$-(\beta \times \mathbf{u}, \nabla \times \hat{\mathbf{u}}) + R_m^{-1}(\nabla \times \mathbf{u}, \nabla \times \hat{\mathbf{u}}) = \lambda(\mathbf{u}, \hat{\mathbf{u}})$$

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Generalisation of Osborn theory for the Pseudo-Spectra

We proved that under uniform converge of the solution operator, the discrete pseudo-spectra converge to the continuous one:
`https://www.uzerbinati.eu/teaching/spectral_theory/`

THANK YOU!

Embedded Trefftz Discretisations for the Maxwell Eigenvalue Problem with Applications to MHD

Umberto Zerbinati*