Material-robust finite elements: strong stress symmetry and angular momentum conservation in mixed formulations

UNIVERSITY OF OXFORD
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Oxford
Mathematics

THE STORY IN ONE SLIDE

Pressure robustness, on the constitutive side

For incompressible flow:

- Discretely div-free schemes admit velocity estimates depending on the pressure.
- Pointwise div-free schemes are **pressure robust**: velocity insensitive to irrotational data perturbations (on simply connected domains).

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This talk

An analogous phenomenon governs mixed discretisations whose constitutive law has a non-trivial kernel. We name this phenomenon **material robustness**.

- Tied to **strong stress symmetry** and **discrete angular momentum conservation**.
- In an abstract framework it reduces to a kernel inclusion $\ker \mathcal{B}_h \subseteq \ker \mathcal{B}$.
- For *anisotropic* laws, weakly symmetric schemes may fail to be robust to perturbations in the kernel of the constitutive relation.

CONTINUUM MECHANICS – BALANCE LAWS

The governing equations of continuum mechanics:

$$
\begin{aligned}
\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) &= 0,\\
\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \nabla\cdot\underline{\underline{\sigma}} &= \rho\boldsymbol{f},\\
\rho(\partial_t \boldsymbol{\eta} + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta}) - \nabla\cdot\underline{\underline{\zeta}} - \boldsymbol{\varsigma} &= \rho\boldsymbol{\tau},
\end{aligned}
$$

with density ρ, velocity $\boldsymbol{u}$, Cauchy stress $\underline{\underline{\sigma}}$, intrinsic angular momentum $\boldsymbol{\eta}$, couple stress $\underline{\underline{\zeta}}$, $\boldsymbol{\varsigma}$ vectorising $\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^T$, body force $\boldsymbol{f}$, body torque $\boldsymbol{\tau}$.

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Closed by **constitutive relations** for $\underline{\underline{\sigma}}$ (and possibly $\underline{\underline{\zeta}}$).

SYMMETRY OF THE STRESS TENSOR AND ANGULAR MOMENTUM

Assume there are no couple stresses or body torques: $\underline{\underline{\zeta}} \equiv 0$, $\boldsymbol{\tau} \equiv 0$. Then

$$\rho\left(\partial_t \boldsymbol{\eta} + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta}\right) = \varsigma,$$

and angular momentum is conserved iff $\varsigma = \mathbf{0}$, i.e. iff $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$.

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Symmetry ⇔ Conservation of angular momentum

Under $\underline{\underline{\zeta}} \equiv 0$, $\boldsymbol{\tau} \equiv 0$, pointwise symmetry of the Cauchy stress is equivalent to the conservation of angular momentum.

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Symmetry $\Leftrightarrow$ Conservation of angular momentum

Under $\underline{\underline{\zeta}} \equiv 0$, $\boldsymbol{\tau} \equiv 0$, pointwise symmetry of the Cauchy stress is equivalent to the conservation of angular momentum.

Discrete corollary. A mixed FE discretisation that produces a pointwise symmetric $\underline{\underline{\sigma}}_h$ exactly conserves discrete angular momentum.

THE HELLINGER–REISSNER FORMULATION

After linearisation about $\boldsymbol{u}=0$ and a linear constitutive law $\varepsilon(\boldsymbol{u})=\frac{1}{2\mu}\underline{\underline{\sigma}}^D+\frac{1}{d(2\mu+d\lambda)}(\operatorname{tr}\underline{\underline{\sigma}})\underline{\underline{I}}-\underline{\underline{F}}$, the saddle-point problem reads: find $\underline{\underline{\sigma}}\in\Sigma^{\mathrm{sym}}$, $\boldsymbol{u}\in V$

$$a(\underline{\underline{\sigma}},\underline{\underline{\tau}})+b(\underline{\underline{\tau}},\boldsymbol{u})=\langle\underline{\underline{\tau}}\boldsymbol{n},\boldsymbol{g}\rangle_{\partial\Omega}+(\underline{\underline{F}},\underline{\underline{\tau}}),\quad b(\underline{\underline{\sigma}},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}).$$

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$\Sigma^{\mathrm{sym}} = H(\mathrm{div}; \Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ encodes strong symmetry in the function space.

STRONGLY VS. WEAKLY SYMMETRIC DISCRETISATIONS

Strong symmetry

$\underline{\underline{\sigma}}_h \in \Sigma_h^{\mathrm{sym}} \subset H(\mathrm{div}; \Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\boldsymbol{u}_h \in V_h$:

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a(\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + b(\underline{\underline{\tau}}_h, \boldsymbol{u}_h) &= \langle \underline{\underline{\tau}}_h \boldsymbol{n}, \boldsymbol{g} \rangle_{\partial\Omega} + (\underline{\underline{F}}, \underline{\underline{\tau}}_h), \quad \forall \underline{\underline{\tau}} \in \Sigma_h^{\mathrm{sym}} \\
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Weak symmetry

$\underline{\underline{\sigma}}_h \in \Sigma_h \subset H(\mathrm{div};\Omega,\mathbb{R}^{d\times d})$, $\boldsymbol{u}_h \in V_h$, multiplier $\underline{\underline{\xi}}_h \in \Xi_h \subset L^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{skw}})$:

$$
\begin{aligned}
&a(\underline{\underline{\sigma}}_h,\underline{\underline{\tau}}_h) + b(\underline{\underline{\tau}}_h,\boldsymbol{u}_h) + c(\underline{\underline{\tau}}_h,\underline{\underline{\xi}}_h) = \langle \underline{\underline{\tau}}_h\boldsymbol{n},\boldsymbol{g}\rangle_{\partial\Omega} + (\underline{\underline{F}},\underline{\underline{\tau}}_h), \quad \forall \underline{\underline{\tau}} \in \Sigma_h \subset H(\mathrm{div};\Omega,\mathbb{R}^{d\times d}),\\
&b(\underline{\underline{\sigma}}_h,\boldsymbol{v}_h) = (\boldsymbol{f},\boldsymbol{v}_h), \quad c(\underline{\underline{\sigma}}_h,\underline{\underline{\omega}}_h) = 0 \quad \forall (\boldsymbol{v}_h,\underline{\underline{\omega}}_h) \in V_h \times \Xi_h.
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SOME WEAKLY SYMMETRIC MIXED FINITE ELEMENTS

Arnold, Brezzi, Douglas (PEERS, 1984); Arnold, Falk, Winther (2007); Amara, Thomas (1979); Stenberg (1988); Cockburn, Gopalakrishnan, Guzman (2010).

PEERS

$$\Sigma_h = \mathcal{RT}_k^{d\times d} \oplus \text{bubbles}, \quad V_h = \mathcal{P}_{k-1}^d, \quad \Xi_h = \mathcal{P}_k \cap H^1.$$

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Familiar $H(\mathrm{div})$ spaces. Available in essentially every FE package.

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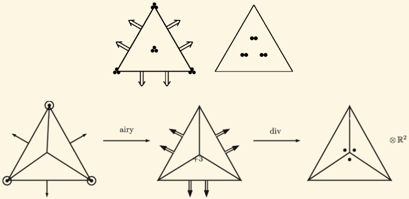

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Hu–Zhang $(\mathrm{HZ}_k,\ k \geq 3)$: symmetric $H(\mathrm{div})$-conforming tensors continuous at vertices.

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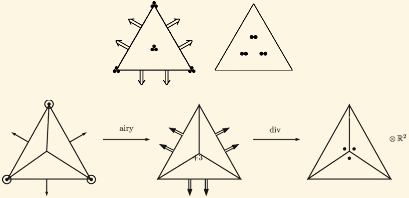
Available in **NGSolve** and **Firedrake**.



DETOUR: INCOMPRESSIBLE STOKES

Stokes flow

$$\underline{\underline{\sigma}} = 2\nu\, \varepsilon(\boldsymbol{u}) - p\underline{\underline{I}}, \qquad \operatorname{div} \boldsymbol{u} = 0.$$
$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \qquad b(\boldsymbol{u}, q) = 0.$$

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Two flavours of discretisation

Discretely div-free: $b(\boldsymbol{u}_h, q_h) = 0\ \forall q_h \in Q_h$ – e.g. Hood–Taylor.

Pointwise div-free: $\operatorname{div} V_h \subseteq Q_h \Rightarrow \operatorname{div} \boldsymbol{u}_h \equiv 0$ – e.g. Scott–Vogelius.

PRESSURE ROBUSTNESS – THE NO-FLOW PROBLEM

🔖 V. John, A. Linke, C. Merdon, M. Neilan, L. Rebholz, **SIREV** 2017.

Pick $\boldsymbol{f} = (0,\ \mathrm{Ra}(1 - y + 3y^2))^T$ so that $\boldsymbol{u} \equiv \boldsymbol{0}$, $p = \mathrm{Ra}(y^3 - \frac{1}{2}y^2 + y - \frac{7}{12})$.

A non-pressure-robust scheme satisfies

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1} \leq C \inf_{\boldsymbol{v}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1} + C(\|\mathrm{div}\, \boldsymbol{u}_h\|_{L^2})\|p - p_h\|_{L^2}.$$

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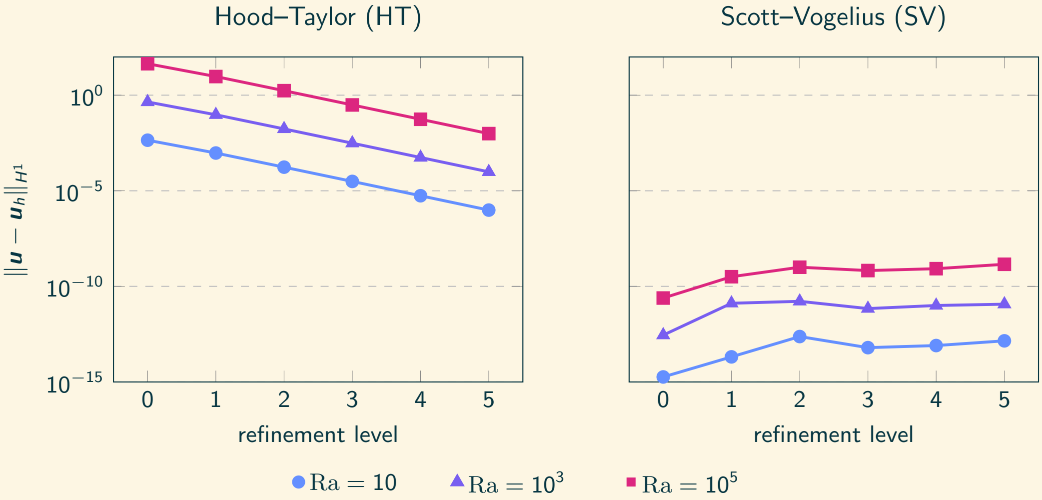
Take-away

For Scott–Vogelius pressure robustness is encoded by $\ker \mathcal{B}_h \subseteq \ker \mathcal{B} = \{\boldsymbol{v} \in V : \operatorname{div} \boldsymbol{v} = 0\}$.

PRESSURE ROBUSTNESS – NUMERICAL EVIDENCE



AN ABSTRACT SADDLE-POINT SETTING

Hilbert spaces $\mathcal{V}$, $\mathcal{Q}$, bounded bilinear forms A, B. Find $u \in \mathcal{V}$, $p \in \mathcal{Q}$:

$$\begin{aligned} A(u, v) + B(v, p) &= F(v) \quad \forall v \in \mathcal{V}, \\ B(u, q) &= G(q) \quad \forall q \in \mathcal{Q}. \end{aligned}$$

Brezzi assumptions hold, i.e. ellipticity in the kernel and the inf-sup condition.

Examples

Stokes: $\mathcal{V} = H^1_0(\Omega)^d$, $\mathcal{Q} = L^2_0(\Omega)$, $B(\boldsymbol{v}, q) = (\operatorname{div} \boldsymbol{v}, q)$.

Strong sym. HR: $\mathcal{V} = \Sigma^{\text{sym}}$, $\mathcal{Q} = V$, $B(\underline{\underline{\tau}}, \boldsymbol{v}) = (\operatorname{div} \underline{\underline{\tau}}, \boldsymbol{v})$.

Weak sym. HR: $\mathcal{V} = \Sigma$, $\mathcal{Q} = V \times \Xi$, $B(\underline{\underline{\tau}}, (\boldsymbol{v}, \underline{\underline{\omega}})) = (\operatorname{div} \underline{\underline{\tau}}, \boldsymbol{v}) + (\underline{\underline{\tau}}, \underline{\underline{\omega}})$.

MATERIAL ROBUSTNESS – THE ENGINEER'S QUESTION

An engineer wants a scheme that works *for any material*: **our job is to make engineer happy!**

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Step 2 – Build a perturbation the constitutive law cannot see

Pick $\boldsymbol{w}$ with $\varepsilon(\boldsymbol{w}) = -\underline{\underline{F}}_{\boldsymbol{w}}$ and $\underline{\underline{\sigma}}_{\boldsymbol{w}} \equiv \underline{\underline{0}}$. Scale by $\alpha \in \mathbb{R}$.

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Step 3 – material robustness

A scheme is **material robust** if its discrete stress error is bounded uniformly in α *across all admissible constitutive laws*:

$$\sup_{\alpha \in \mathbb{R}} \|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h^{(\alpha)}\|_{H(\mathrm{div})} < \infty.$$

Robustness of the scheme *independently of the material* – the elasticity-side counterpart of pressure robustness.

MATERIAL ROBUSTNESS VIA A KERNEL INCLUSION

The perturbation enters the variational problem as $F \mapsto F + B(\cdot, r)$, with $r \in \mathcal{Q}$ encoding $\boldsymbol{w}$ (and possibly $\underline{\underline{\omega}} = \operatorname{skw} \nabla \boldsymbol{w}$).

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$u_{h,r} = u_h$ for all $r \in \mathcal{Q} \Longleftrightarrow \boxed{\ker \mathcal{B}_h \subseteq \ker \mathcal{B}}$.

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Material robustness $=$ kernel-inclusion principle, applied to $B(\underline{\underline{\tau}}, (\boldsymbol{v}, \underline{\underline{\omega}}))$ on the stress formulation.

PATCH TEST 1 – RIGID BODY MOTION (ISOTROPIC)

Linear isotropic law: $\underline{\underline{F}} \equiv 0$. Kernel of $\boldsymbol{u} \mapsto \varepsilon(\boldsymbol{u})$ = **rigid body motions**.

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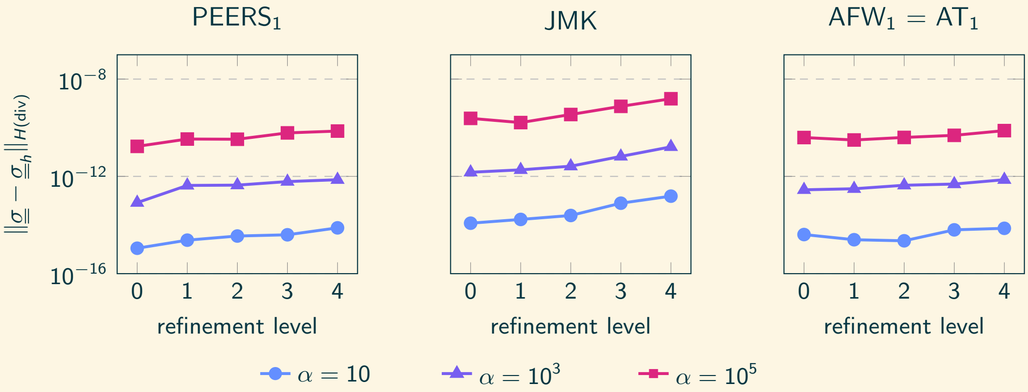
Any failure to recover $\underline{\underline{\sigma}} \equiv 0$ as α grows is a **structural** failure of the scheme.

PATCH TEST 1 – NUMERICAL RESULTS



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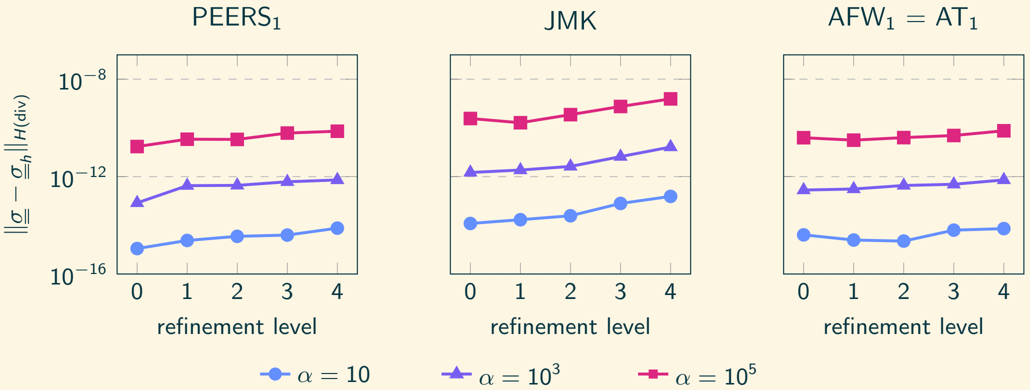

All three are material robust on this test. **Why?**

WHY WEAK SYMMETRY SUFFICES FOR AN ISOTROPIC CONSTITUTIVE LAW

For an *isotropic* constitutive law, **frame indifference** of the energy functional forces the kernel of $\boldsymbol{w} \mapsto \underline{\underline{\sigma}}(\boldsymbol{w})$ to consist exactly of the **rigid body motions**.

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Rigid body motions are linear vector fields, in particular contained in $\mathcal{RT}_1(\mathcal{T}_h)^d$. Their skew gradient is constant, so

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Hence

It suffices that Ξ_h capture constant skew tensors, i.e. $\mathrm{DG}_0(\mathsf{skw}) \subset \Xi_h$. Every reasonable weakly symmetric element (PEERS, AFW$_k$, …) satisfies this. The issue went unnoticed because the literature is dominated by isotropic constitutive laws.

LIQUID CRYSTAL POLYMER NETWORKS – TRANSVERSE ANISOTROPY

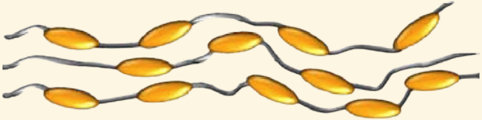
T. J. White, **J. Polymer Sci.** 2017; R. H. Nochetto et al., **SINUM** 2023.

Liquid crystal polymer networks (LCNs): cross-linked **polymers** with a **liquid-crystalline** phase. Prototype: **Kevlar**.

Transversely isotropic constitutive law

$$\underline{\underline{\sigma}} = 2\mu\,\varepsilon(\boldsymbol{u}) + \lambda(\operatorname{div}\boldsymbol{u})\underline{\underline{I}} + \alpha\,\boldsymbol{\nu}\otimes\boldsymbol{\nu}.$$

Director $\boldsymbol{\nu}$ encodes the preferred direction of the polymer chains.



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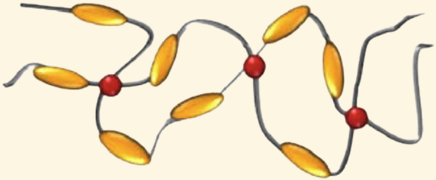


LIQUID CRYSTAL POLYMER NETWORKS – TRANSVERSE ANISOTROPY

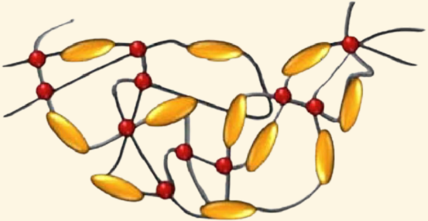
T. J. White, **J. Polymer Sci.** 2017; R. H. Nochetto et al., **SINUM** 2023.

Liquid crystal polymer networks (LCNs): cross-linked **polymers** with a **liquid-crystalline** phase. Prototype: **Kevlar**.

Transversely isotropic constitutive law

$$\underline{\underline{\sigma}} = 2\mu\, \varepsilon(\boldsymbol{u}) + \lambda(\mathrm{div}\, \boldsymbol{u})\underline{\underline{I}} + \alpha\, \boldsymbol{\nu} \otimes \boldsymbol{\nu}.$$

Director $\boldsymbol{\nu}$ encodes the preferred direction of the polymer chains.



PATCH TEST 2 – TRANSVERSE ANISOTROPY

Set $\underline{\underline{F}} = \alpha\, \boldsymbol{\nu} \otimes \boldsymbol{\nu}$. The kernel is $\{\boldsymbol{w} : \varepsilon(\boldsymbol{w}) = -\underline{\underline{F}}\}$.

Saint–Venant compatibility

On a contractible domain, solvable iff $\nabla^T \times (\nabla \times \underline{\underline{F}}) = \underline{\underline{0}}$.

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For a **polynomial** director $\boldsymbol{\nu}(x, y) = (x, x + y)^T$, integration gives a polynomial $\boldsymbol{w}$ with

$$\underline{\underline{\omega}}_{12} = \frac{1}{2}(\partial_x w_2 - \partial_y w_1) = \frac{\alpha}{2\mu}(2xy + 2y^2 + x^2) \in \mathcal{P}_2.$$

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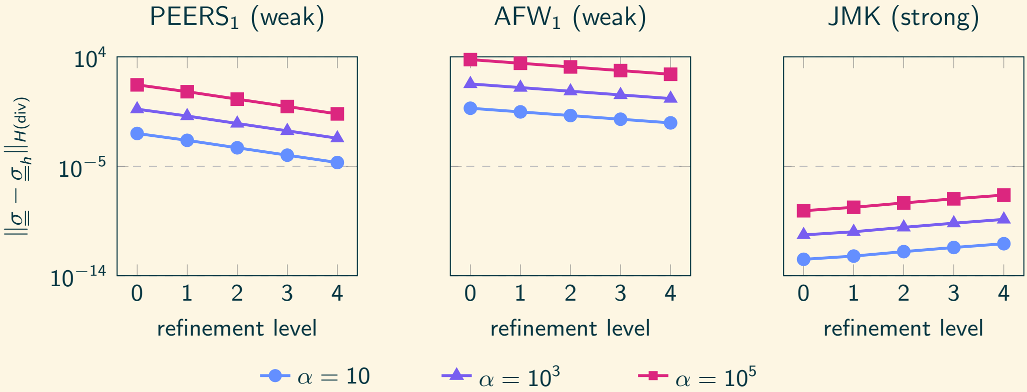
Caveat. This characterisation only holds for *polynomial* solutions of $\varepsilon(\boldsymbol{w}) = -\underline{\underline{F}}$. For non-polynomial directors $\boldsymbol{w}$ leaves the polynomial space and $\underline{\underline{\omega}} \notin \Xi_h$ for any finite k.

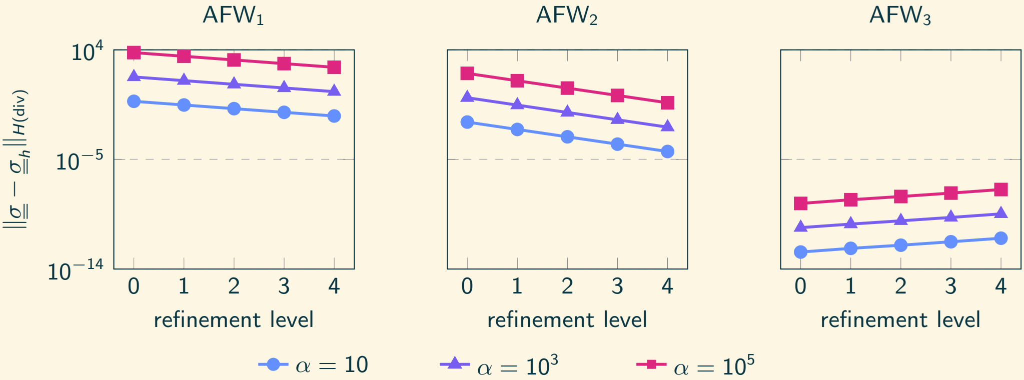
PATCH TEST 2 – NUMERICAL RESULTS, LOW ORDER



Weakly symmetric schemes lose material robustness: the stress error scales like α. JMK does not.

HIGH ORDER CAN SAVE WEAK SYMMETRY

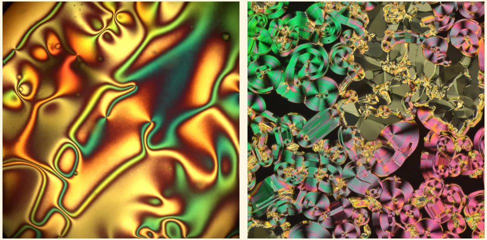
AFW_k has $\Xi_h = \mathcal{P}_{k-1}(\mathcal{T}_h)$; the exact $\underline{\underline{\omega}}$ here is a polynomial of degree 2.



Robustness is recovered when Ξ_h contains $\underline{\underline{\omega}}$. **Recovery is conditional on the data.**

POLAR FLUIDS – THE ERICKSEN STRESS TENSOR

J. L. Ericksen, **Trans. Soc. Rheology** 1961; F. M. Leslie, **Quart. J. Mech.** 1966.



Polar fluid constitutive law

$\underline{\underline{\sigma}} = 2\mu\,\varepsilon(\boldsymbol{u}) - p\underline{\underline{I}} + K_F\,\nabla\boldsymbol{\nu}^T\nabla\boldsymbol{\nu}$, with div $\boldsymbol{u} = g_{\text{div}}$, $\boldsymbol{\nu}$ the director, K_F the Frank constant.

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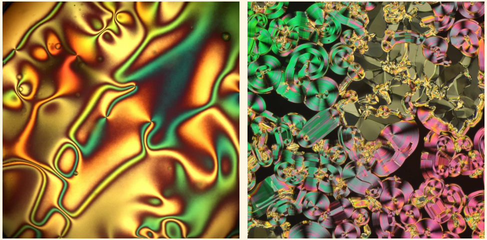

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Eliminating p we are in our framework with $\lambda = \infty$ and $\underline{\underline{F}} = \frac{1}{2\mu}(K_F\nabla\boldsymbol{\nu}^T\nabla\boldsymbol{\nu})^D - \frac{1}{d}g_{\text{div}}\underline{\underline{I}}$.

POLAR FLUIDS IN 2D – WORST CASE

Set $\boldsymbol{\nu}(x,y) = (x,y)^T$ so that $\nabla\boldsymbol{\nu}^T\nabla\boldsymbol{\nu} = \underline{\underline{I}}$. Choose the velocity, pressure and Frank constant

$$\boldsymbol{u}(x,y) = -\frac{\alpha}{\mu}\begin{pmatrix} -\cos x \cosh y \\ \sin x \sinh y \end{pmatrix}, \quad p = -\alpha \sin x \cosh y, \quad K_F = \alpha \sin x \cosh y.$$

Direct computation: $\underline{\underline{\sigma}} \equiv 0$ for every α, but

$$\underline{\underline{\omega}} = -\frac{\alpha}{\mu}\begin{pmatrix} 0 & -\cos x \sinh y \\ \cos x \sinh y & 0 \end{pmatrix}.$$

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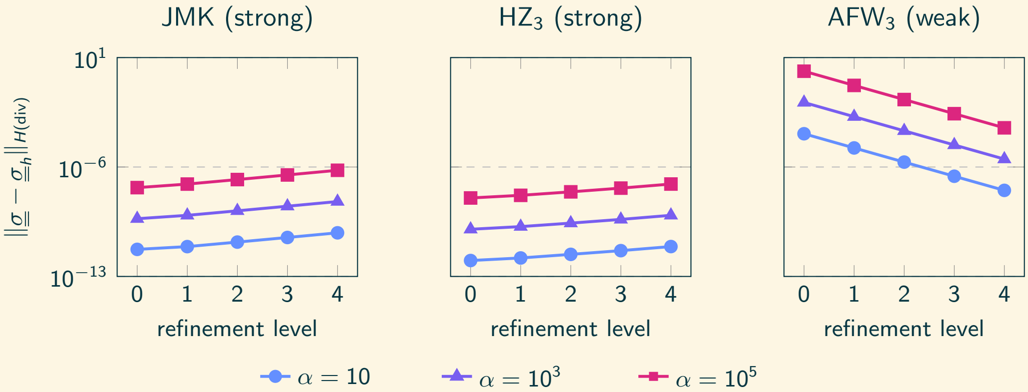
$$\underline{\underline{\omega}} = -\frac{\alpha}{\mu}\begin{pmatrix} 0 & -\cos x \sinh y \\ \cos x \sinh y & 0 \end{pmatrix}.$$

The exact rotation tensor $\underline{\underline{\omega}}$ is **not** a polynomial: $\underline{\underline{\omega}} \notin \Xi_h$ for *any* piecewise polynomial Ξ_h.
Increasing the polynomial degree cannot save weak symmetry here.

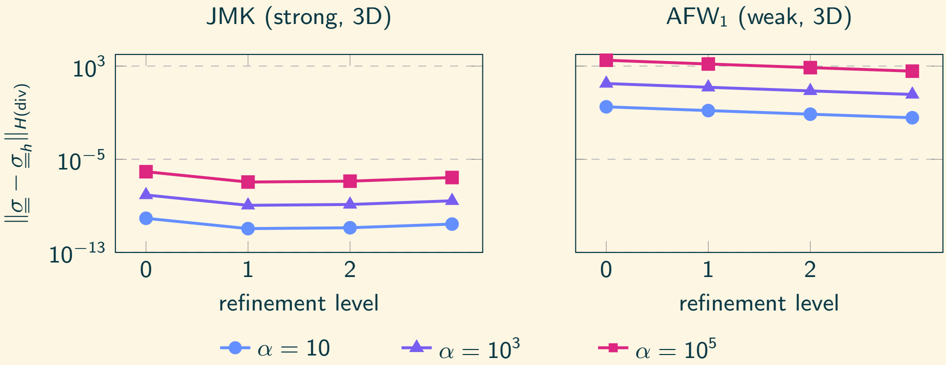
POLAR FLUIDS 2D – NUMERICAL RESULTS



Strongly symmetric (JMK, HZ$_3$): stress error at solver tolerance. Weakly symmetric (AFW$_3$): error scales like α – material robustness fails irrecoverably.

POLAR FLUIDS IN 3D

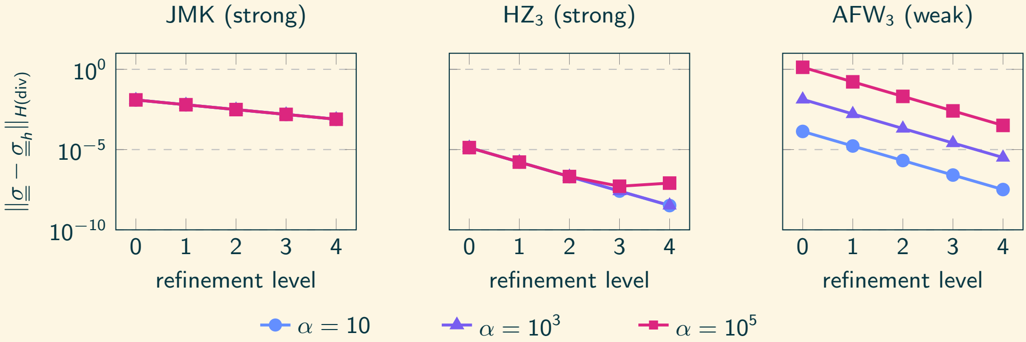
Divergence-free director $\boldsymbol{\nu} = (x + \sin y,\ y,\ z)^T$ with $\nabla\boldsymbol{\nu}^T\nabla\boldsymbol{\nu}$ satisfying Saint–Venant; $K_F = \alpha$.



Same dichotomy as in 2D: JMK preserves angular momentum; AFW$_1$ does not.

BEYOND STRESS-FREE: A NON-TRIVIAL REFERENCE SOLUTION

Same polar problem in 2D, but now we add a non-kernel component to the exact solution: $\underline{\underline{\sigma}} \neq \underline{\underline{0}}$. The kernel component is still scaled by α.



Strongly symmetric schemes converge optimally and are insensitive to α. The weakly symmetric error still scales like α on the kernel component, even though the exact stress is non-trivial.

WHERE WE STAND

Empirical pattern across the four examples

Strongly symmetric schemes are **always** material robust.

Weakly symmetric schemes are material robust *iff* the exact rotation $\underline{\underline{\omega}} = \operatorname{skw} \nabla \boldsymbol{w}$ lies in Ξ_h:

Isotropic constitutive law: $\underline{\underline{\omega}} =$ const skew $\in \Xi_h$ for every standard element.
Transversely isotropic, polynomial director: $\underline{\underline{\omega}} \in \mathcal{P}_2$, so $k \geq 3$ recovers it.
Polar fluid: $\underline{\underline{\omega}}$ transcendental $\Rightarrow$ no polynomial Ξ_h suffices.
Non-stress-free: optimal convergence preserved by strong sym.; weak symmetry still polluted by the α-component.

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Goal

A general theory that explains all four observations – and unifies them with pressure robustness.

THE CONTINUOUS INVARIANCE LEMMA

Lemma (Saddle-point invariance)

Fix $r \in \mathcal{Q}$. Let u_r, p_r solve

$$A(u_r, v) + B(v, p_r) = F(v) + B(v, r) \quad \forall v \in \mathcal{V},$$
$$B(u_r, q) = G(q) \quad \forall q \in \mathcal{Q}.$$

Then $\boxed{u_r = u, \quad p_r = p + r}$.

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Then $\boxed{u_r = u, \quad p_r = p + r}$.

The primary variable does not see r. Stress is unaffected by perturbations within the kernel of the constitutive law – analogue of the Stokes velocity not seeing $\nabla\phi$.

EACH EXAMPLE FITS THE ABSTRACT PERTURBATION FORM

Each constitutive scenario in the talk is a special case of the saddle-point shift $F \mapsto F + B(\cdot, r)$, with r derived from a kinematic field $\boldsymbol{w}$ in the kernel of the constitutive map (Saint–Venant + integration):

Linear isotropic: $\underline{\underline{F}} \equiv \underline{\underline{0}} \Rightarrow \boldsymbol{w}$ is a rigid body motion.

Transversely isotropic: $\underline{\underline{F}} = \alpha\, \boldsymbol{\nu} \otimes \boldsymbol{\nu}$, choose $\boldsymbol{\nu}$ so that Saint–Venant's compatibility holds; $\boldsymbol{w}$ is integrated polynomial.

Polar fluid: $\underline{\underline{F}} = \frac{1}{2\mu}(K_F \nabla \boldsymbol{\nu}^T \nabla \boldsymbol{\nu})^D - \frac{g_{\mathrm{div}}}{d} \underline{\underline{I}}$; choose $\boldsymbol{\nu}$, K_F, g_{div} so that the kinematic kernel is non-empty.

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Constructing such examples is a recipe for stress-testing weak-symmetry: pick any $\underline{\underline{F}}$ for which Saint–Venant integrates to a non-polynomial $\boldsymbol{w}$ and you have an obstruction to material robustness for any AFW_k.

TWO NOTIONS OF STRUCTURE PRESERVATION – AND THEY COINCIDE

Discrete problem on $\mathcal{V}_h \times \mathcal{Q}_h$, perturbed solution $u_{h,r}, p_{h,r}$.

Notion 1: Fundamental invariance

The discrete scheme is structure preserving if $u_{h,r} = u_h$ for every $r \in \mathcal{Q}$.

Notion 2: Kernel inclusion

The discrete kernel is contained in the continuous kernel: $\ker \mathcal{B}_h \subseteq \ker \mathcal{B}$.

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Theorem

Under the Babuska–Brezzi assumptions, the two notions are equivalent:

$$u_{h,r} = u_h \;\; \forall r \in \mathcal{Q} \quad \Longleftrightarrow \quad \ker \mathcal{B}_h \subseteq \ker \mathcal{B}.$$

THE PRESSURE / MATERIAL ROBUSTNESS DICTIONARY

	Stokes / pressure robust.	**HR / material robust.**
$\mathcal{V}$	$H^1_0(\Omega)^d$	Σ^{sym} (or Σ)
$\mathcal{Q}$	$L^2_0(\Omega)$	V (or $V \times \Xi$)
$B(\cdot,\cdot)$	$(\mathrm{div}\, \boldsymbol{v}, q)$	$(\mathrm{div}\, \underline{\underline{\tau}}, \boldsymbol{v})$
$\ker \mathcal{B}$	div-free fields	rigid-stress kinematics
Robust scheme	pointwise div-free V_h	strongly sym. Σ_h^{sym}
Patch problem	no-flow, $\boldsymbol{f} = \nabla\phi$	RBM / trans. iso / polar
Failure mode	vel. error $\sim \|p\|$	stress error $\sim \alpha$

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One unified principle: $\ker \mathcal{B}_h \subseteq \ker \mathcal{B}$. Pressure robustness and material robustness are siblings.

A PRIORI ERROR ESTIMATES

Under $\ker \mathcal{B}_h \subseteq \ker \mathcal{B}$ there is a unique linear projection $\Phi_h : \mathcal{Q} \to \mathcal{Q}_h$ with $B(v_h, q - \Phi_h q) = 0$ for all $v_h \in \mathcal{V}_h, q \in \mathcal{Q}$.

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Robust quasi-optimality

For structure-preserving schemes, the primary-variable error decouples from the multiplier:

$$\|u - u_h\|_{\mathcal{V}} \leq C \inf_{v_h \in \mathcal{V}_h} \|u - v_h\|_{\mathcal{V}}.$$

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For non-structure-preserving schemes, the multiplier can pollute the primary variable – and the patch tests realise this obstruction.

A TRANSIENT EXAMPLE – TRANSVERSELY ISOTROPIC FLUID

Lid-driven cavity for an incompressible polar fluid, anisotropy frozen in time:

$$\partial_t \boldsymbol{u} - \operatorname{div} \underline{\underline{\sigma}} = 0, \quad \underline{\underline{\sigma}} = 2\mu\, \varepsilon(\boldsymbol{u}) - p\underline{\underline{I}} + \alpha K_F \nabla \boldsymbol{\nu}^T \nabla \boldsymbol{\nu},$$

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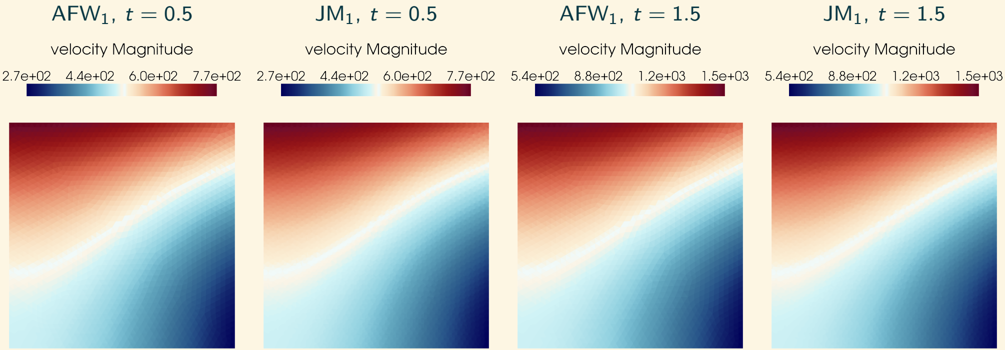
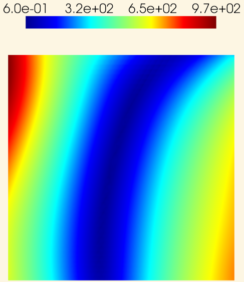
Boundary data drives the solution towards a near-kernel flow. Compared schemes: JM_1 (strong sym.), AFW_1 (weak sym.), SV_4 (pressure-robust primal, no symmetry). Radau IIA, $\Delta t = 0.01$.

TRANSIENT – VELOCITY, AFW$_1$ VS. JM$_1$ AT $t = 0.5$ AND $t = 1.5$

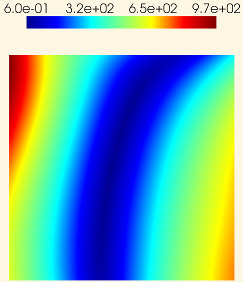


TRANSIENT – STRESS, AFW$_1$ VS. JM$_1$ AT $t = 0.5$ AND $t = 1.5$

AFW$_1$, $t = 0.5$



JM$_1$, $t = 0.5$



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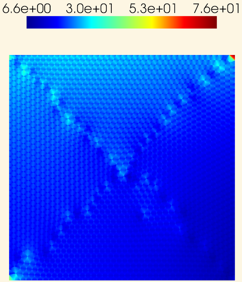

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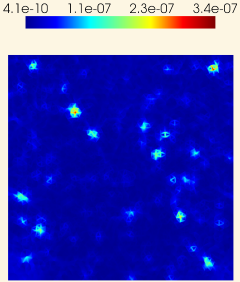

TRANSIENT – WHAT WE OBSERVE

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Material robustness in time

The time-dependent setting is outside our static theory, but the kernel inclusion still governs leading-order behaviour. Non-robustness errors *accumulate*.

ANTISYMMETRIC STRESS AND ARTIFICIAL TORQUE

Recall: in the absence of body torques and couple stresses,

$$\rho(\partial_t \boldsymbol{\eta} + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta}) \;=\; \varsigma, \qquad \varsigma \sim \text{vectorised skw}\, \underline{\underline{\sigma}}.$$

A non-zero antisymmetric component of $\underline{\underline{\sigma}}_h$ acts as an **artificial torque** on the discrete continuum.

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Beyond the one-constant approximation

In liquid crystal theory, the standard *one-constant* Oseen–Frank energy yields a symmetric Cauchy stress. For the *full* Oseen–Frank energy, $\underline{\underline{\sigma}}$ is genuinely **non-symmetric** and the antisymmetric component carries physical couple stresses.

Then we must model couple stresses explicitly, and the discrete antisymmetric component must be the right one – a much sharper requirement than current weakly symmetric schemes can deliver.

FUTURE WORK

Strongly symmetric MCS on Alfeld splits, on curved elements

With J. Gopalakrishnan, P. L. Lederer, J. Schoberl. Performing an Alfeld split of the MCS scheme makes it strongly symmetric and the construction extends to **curved elements** – a setting where most existing strongly symmetric tensor elements are not yet available.

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TDNNS–Nedelec mixed stress without pressure

With E. Zampa, M. Neunteufel. A mixed stress formulation using TDNNS elements for $\underline{\underline{\sigma}}$ and Nedelec elements for $\boldsymbol{u}$. Eliminating the pressure removes the h-dependence of the inf-sup constant observed in Boon–Hiptmair–Tonnon–Zampa, while preserving strong symmetry.

TAKE-AWAYS

Material Robustness

Propose the idea of **material robustness** – numerical scheme should be robust independently on the material decide to simulate, i.e. on the constitutive relation of the model.

Identified **strong stress symmetry** and therefore also **angular momentum conservation** as the structural property that secures it.

Unified with the theory of pressure robustness via $\ker \mathcal{B}_h \subseteq \ker \mathcal{B}$.

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Practical message

For *anisotropic* continuum models, weakly symmetric schemes can fail to be material robust even at high order. Strongly symmetric tensor elements (NGSolve, Firedrake) should be the default when the constitutive law admits a non-trivial kernel.

WHERE TO LOOK

This talk: P. Brubeck, C. Parker, U. Zerbinati, *Achieving Material Robustness via Symmetric Stress Finite Element Discretizations*, in preparation (2026).

Pressure robustness: V. John, A. Linke, C. Merdon, M. Neilan, L. Rebholz, *On the divergence constraint in mixed FE methods for incompressible flows*, SIAM Review (2017).

Symmetric tensor elements: D. N. Arnold, R. Winther (2002); J. Hu, S. Zhang (2015); C. Johnson, B. Mercier (1978); J. Gopalakrishnan, J. Guzman, J.J. Lee (2025).

Software: P. D. Brubeck, R. C. Kirby (2025) – Firedrake; J. Schoberl – NGSolve.

Thank you!

THANK YOU!

Material-robust finite elements:
strong stress symmetry and angular momentum conservation in mixed formulations

Pablo Brubeck*, Charles Parker**, Umberto Zerbinati*