The lightning VEM for eigenvalue problems

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A self-adjoint eigenvalue problem

We here consider a prototypical self-adjoint eigenvalue problem:

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We will also discuss a similar eigenvalue problem for the **resonance** of an elastic beam.

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$$\mathbb{B}_k(\partial K) := \left\{ v_h \in \mathcal{C}^0(\partial K) \ : \ \forall e \in \partial K,\ v_h|_e \in \mathbb{P}_k(e) \right\}.$$

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Do we really need to solve the Poisson problem to solve the Poisson problem ?

The projectors

To avoid the **"recursive"** solution of the Poisson problem, we introduce the following projectors:

$$\Pi_k^{\nabla,K} : V_h^k(K) \to \mathbb{P}_k(K)$$

$$\begin{cases} \displaystyle\int_K \nabla p_k \cdot \nabla(v_h - \Pi_k^{\nabla,K} v_h)\,\mathrm{d}K = 0 \quad \forall v_h \in V_h^k(K) \;\; \forall p_k \in \mathbb{P}_k(K), \\ \displaystyle\int_{\partial K} (v_h - \Pi_k^{\nabla,K} v_h)\,\mathrm{d}s = 0\,. \end{cases}$$

The discretisation

Using the projectors, we can discretise the original eigenvalue problem as:

$$\sum_{K\in\mathcal{T}_h} a^K(u,v) = \sigma \sum_{K\in\mathcal{T}_h} b^K(u,v), \qquad \forall v \in [\hat{V}_h(\mathcal{T}_h)]^d.$$

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Where the bilinear forms $a^K(\cdot, \cdot)$ and $b^K(\cdot, \cdot)$ are defined as:

$$\begin{aligned} a_h^K(u_h, v_h) := &(\nabla \Pi_k^{\nabla, K} u_h, \nabla \Pi_k^{\nabla, K} v_h)_{0,\Omega} \\ &+ \alpha\, \mathcal{S}^K((I - \Pi_k^{\nabla, K}) u_h, (I - \Pi_k^{\nabla, K}) v_h) \end{aligned}$$

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$$\begin{aligned} b_h^K(u_h, v_h) &:= (\Pi_k^K u_h, \Pi_k^K v_h)_{0,\Omega} \\ &\quad + \beta\, \mathcal{S}^K((I-\Pi_k^K)u_h, (I-\Pi_k^K)v_h) \end{aligned}$$

The degress of freedom

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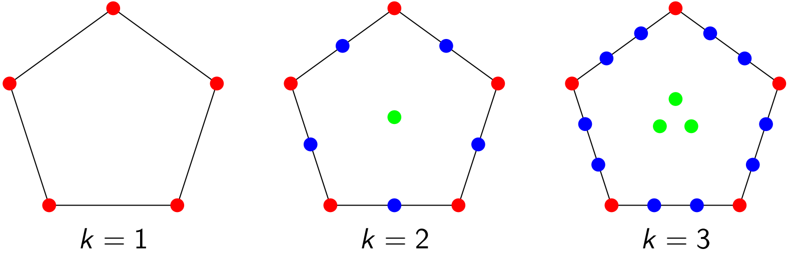
Notice that to construct the projectors associated with the VEM space, we only need a few degrees of freedom more than the one needed to construct the FEM space.



Degrees of freedom on a pentagon, for $k = 1, 2, 3$.

The stabilisation

Where the stabilisation term $\mathcal{S}^K(\cdot,\cdot)$ needs to resepct the following properties:

$$C_*|v_h|^2_{1,E} \leq \mathcal{S}(v_h, v_h) \leq C^*|v_h|^2_{1,E}\,, \qquad \text{for all } v_h \in \mathrm{Ker}(\Pi_k^{\nabla,K})\,,$$

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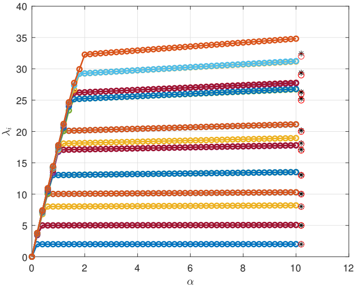
The **jinx of stabilisation** comes out swinging:

- How to choose the correct stabilisation term, given the lack of physical intuition behind it ?
- The stabilisation term might note be **robust** with respect to the polynomial degree k.
- The stabilisation term has a **polluting** effect on the spectrum.

The polluting effect of the stabilisation

Let us focus on the last form of the jinx of stabilisation.



We computed the eigenvalues of for different α and fixed stabilization $\beta = 5$. The horizontal line represents the "good" eigenvalues, while the oblique line represents the "spurious" eigenvalues. The red circle reppresents the exact eigenvalues.

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- In *Berrone et all. (2024)*, the idea is extended by projecting on a space of higher order divergence-free polynomials.
- In *Berrrone et all. (2023)*, the authors propose to approximate the VEM basis functions by the use of a neural network. This approach brings the VEM method back into the realm of classical finite elements.

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We aim to explicitly construct basis functions for the VEM space.

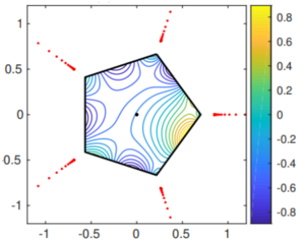
- This will allow treating the VEM as a classical finite element method, hence removing the need for any stabilisation.
- We will have point-wise access to the value of the basis functions, and therefore also of the solution.

The basis functions

We consider as basis functions a lightning approximation of the exact solution of the Poisson problem, inside each element, i.e.

$$\hat{\phi}_i = \mathrm{Re}\left\{ \sum_{j=0}^{N_P} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_Z} b_j z^j \right\},$$

where $\{z_j\}_{j=0}^{N_P}$ are poles clustered exponentially close to the vertices of the polygon.



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- The lightning VEM is a slightly **non-conforming** finite element scheme, hence we are committing a variational crime.
- We need to be careful with the quadrature use we decided to use, so far **high-order Gauss-Lobatto quadrature** did the trick.

Results

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Table: Eigenvalues of the vibrating membrane problem, computed using the lightning VEM, for different number of elements N.

	Computed (rate)			
Exact	$N = 16$	$N = 64$	$N = 256$	$N = 1024$
2	2.1041 (-)	2.0272 (2.3)	2.0069 (1.9)	2.0016 (2.2)
5	5.7076 (-)	5.1704 (2.5)	5.0420 (2.0)	5.0104 (2.1)
5	5.7827 (-)	5.1766 (2.6)	5.0441 (1.9)	5.0107 (2.2)
8	9.6766 (-)	8.4257 (2.4)	8.1095 (1.9)	8.0274 (2.1)
10	13.0473 (-)	10.6908 (2.6)	10.1675 (2.0)	10.0418 (2.1)
10	13.1213 (-)	10.7088 (2.6)	10.1774 (1.9)	10.0430 (2.2)
13	17.2137 (-)	14.1258 (2.3)	13.2883 (1.9)	13.0709 (2.2)
13	17.4080 (-)	14.1523 (2.4)	13.2904 (1.9)	13.0743 (2.1)
17	25.0125 (-)	19.0137 (2.5)	17.4854 (2.0)	17.1218 (2.1)
17	32.9151 (-)	19.0700 (3.6)	17.5246 (1.9)	17.1242 (2.2)

The vibrating beam

We also consider the eigenvalue problem associated with a vibrating elastic beam, i.e.

$$\nabla \cdot \left(2\mu\,\varepsilon(\boldsymbol{u}) + \lambda\,\mathrm{div}(\boldsymbol{u})I\right) = \sigma\boldsymbol{u},$$

where ε is the symmetric gradient, here used as a classical measure of strain.

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As λ tends to infinity the elastic beam becomes more and more incompressible, while μ is a parameter mainly describing the deviatoric response of the beam.

Results for the virabting beam

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Table: Eigenvalues for fixed Young's modulus $E = 70$ and Poisson's ratio $\nu = 0.2$. The reference eigenvalues are computed using a high-order FEM.

	Computed (rate)			
Reference	$N = 16$	$N = 64$	$N = 256$	$N = 1024$
1007.87	1079.25 (-)	1026.15 (2.4)	1012.43 (1.9)	1009.00 (2.1)
1007.87	1081.55 (-)	1026.34 (2.5)	1012.54 (1.9)	1009.02 (2.2)
1492.37	1833.06 (-)	1576.01 (2.5)	1513.10 (2.0)	1497.41 (2.2)
2165.99	2504.26 (-)	2263.84 (2.2)	2191.22 (1.9)	2172.37 (2.1)
2755.46	3405.34 (-)	2907.61 (2.6)	2794.34 (1.9)	2765.06 (2.2)
2882.72	4032.00 (-)	3172.83 (2.4)	2954.98 (1.9)	2900.39 (2.2)
2882.72	4086.72 (-)	3177.66 (2.5)	2956.80 (1.9)	2900.69 (2.2)
3529.69	4146.86 (-)	3664.37 (2.7)	3563.53 (1.9)	3538.40 (2.1)
4082.41	5146.52 (-)	4430.68 (1.9)	4175.24 (1.8)	4105.75 (2.1)
4082.41	5368.48 (-)	4439.26 (2.3)	4176.30 (1.9)	4106.08 (2.1)

Locking

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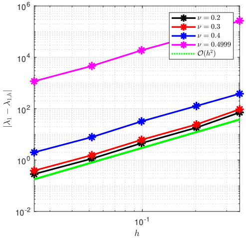
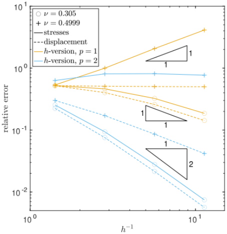
It is a well-known issue of low-order discretisation of linear elasticity, that for $\lambda \to \infty$, the discretisation converge to the wrong answer. This phenomenon is known as **locking**.





The figure on the left comes is taken from *Ainsworth and Parker (2022)*.

Thank you for you attention !