Discretisation of the Helmholtz–Korteweg equation

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Mathematical Modelling, Numerical Analysis and Scientific Computing, Kácov, 18th June 2025

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L. S. Hirst, G. Charras, *Liquid crystals in living tissue*, Nature, 2017.

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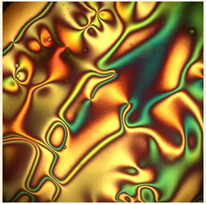
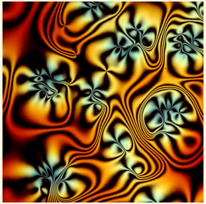
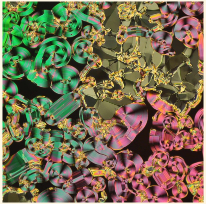
LIQUID CRYSTALS

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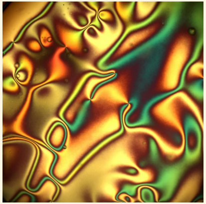
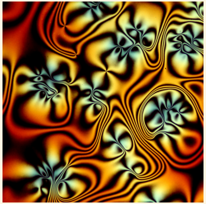
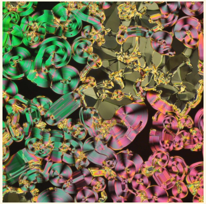
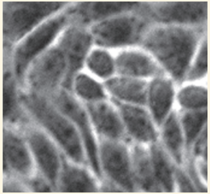
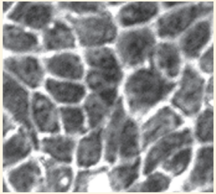

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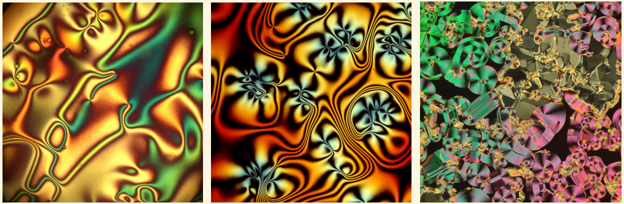

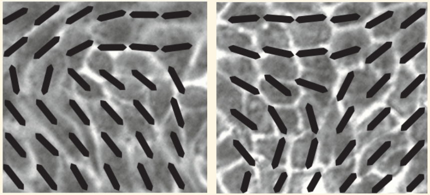

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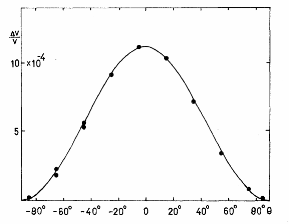

Figure: Angular dependence of sound velocity. $T = 21$ C, $v = 10$ MHz, and $H = 5$ kOe. θ is the angle between the field direction and propagation direction. Solid line is $12.5 \cdot 10^{-4} \cos(\theta)^2$.

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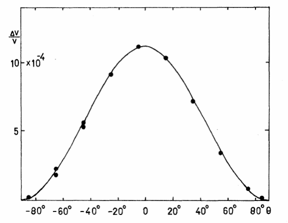

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- Historically the interaction of acoustic waves with the nematic director field was first explained by means of the minimal entropy production principle.
- We here assume the aligning torque acting on the nematic director field is of elastic nature, rather than of a dissipative viscous one. This idea was already proposed, and validated experimentally, by Mullen, Lüthi, and Stephen.

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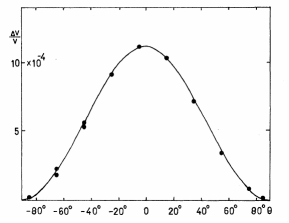

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THE NEMATIC HELMHOLTZ–KORTEWEG EQUATION

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Under this hypothesis the **nematic Helmholtz–Korteweg equation** can be derived, i.e.

$$-\omega^2 S(\boldsymbol{x}) - c_0^2 \Delta S(\boldsymbol{x}) + \rho_0^2 \alpha \Delta^2 S(\boldsymbol{x}) + \rho_0^2 u_2 \nabla \cdot \nabla \left[\boldsymbol{n} \cdot \underline{\underline{\mathcal{H} S}} \boldsymbol{n}\right] = 0.$$

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- It can be proven that this PDE is compatible with a hyperelastic formulation, i.e. it can be derived from the free energy functional
$$W(\rho, \nabla \rho, \boldsymbol{n}) = c_0^2 \rho^2 + \frac{1}{2}\alpha \|\nabla \rho\|^2 + \frac{1}{2}\beta(\nabla \rho \cdot \boldsymbol{n})^2.$$

BOUNDARY CONDITIONS

Mathematical Institute

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Sound-soft boundary conditions

Sound-soft boundary conditions impose that the excess-pressure defined as

$$c_0^2\rho_0 S(\boldsymbol{x}) - \rho_0^3\alpha\Delta S(\boldsymbol{x}) - u_2\rho_0^3\left(\boldsymbol{n}\cdot\underline{\underline{\mathcal{H}S}}\boldsymbol{n}\right) = 0.$$

vanishes along the boundary. Sound-soft boundary conditions thus correspond to imposing homogeneous Dirichlet boundary conditions on $S(\boldsymbol{x})$ and

$$\Delta S(\boldsymbol{x}) = -\frac{u_2}{\alpha}\left(\boldsymbol{n}\cdot\underline{\underline{\mathcal{H}S}}\boldsymbol{n}\right).$$

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Sound-hard boundary conditions

Sound-hard boundary conditions also change since the normal derivative of the fluid velocity $\partial_{\boldsymbol{\nu}} \boldsymbol{v}$ now satisfies the equation

$$i\omega\rho_0(\boldsymbol{n}\cdot\boldsymbol{\nu}) = c_0^2\partial_{\boldsymbol{\nu}} S(\boldsymbol{x}) - \rho_0^2\alpha\partial_{\boldsymbol{\nu}}\Delta S(\boldsymbol{x}) - \rho_0^2 u_2\partial_{\boldsymbol{\nu}}\left(\boldsymbol{n}\cdot\underline{\underline{\mathcal{H}S}}\boldsymbol{n}\right).$$

Sound-hard boundary conditions thus correspond to imposing homogeneous Neumann boundary conditions on $S(\boldsymbol{x})$ and

$$\partial_{\boldsymbol{\nu}}\Delta S(\boldsymbol{x}) = -\frac{u_2}{\alpha}\partial_{\boldsymbol{\nu}}\left(\boldsymbol{n}\cdot\underline{\underline{\mathcal{H}S}}\boldsymbol{n}\right).$$

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Impedance boundary conditions

Some computation shows that the impedance boundary conditions for the nematic Helmholtz–Korteweg equation are equivalent to imposing Robin boundary conditions on $S(\boldsymbol{x})$ and

$$\partial_{\boldsymbol{\nu}} \Delta S(\boldsymbol{x}) = i\zeta \Delta S(\boldsymbol{x}) + i\zeta \frac{u_2}{\alpha} \left(\boldsymbol{n} \cdot \underline{\underline{\mathcal{H}S}}\boldsymbol{n}\right) - \frac{u_2}{\alpha} \partial_{\boldsymbol{\nu}} \left(\boldsymbol{n} \cdot \underline{\underline{\mathcal{H}S}}\boldsymbol{n}\right),$$

where ζ is the impedance of the boundary.

WEAK FORMULATION

Farrell, P.E., van Beeck, T. and ∼., 2025. Analysis and numerical analysis of the Helmholtz–Korteweg equation. arXiv preprint arXiv:2503.10771.

We want to find $u \in X$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in X,$$

where

$$a(u, v) := \underbrace{\alpha(\Delta u, \Delta v)_{L^2(\Omega)} + \beta(\boldsymbol{n}^T(\mathcal{H}u)\boldsymbol{n}, \Delta v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}}_{=:e(u,v)} - k^2(u, v)_{L^2(\Omega)}$$

We only consider sound-soft boundary conditions for which $X = H_0^2(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$ for simplicity and impose the boundary conditions using Nitsch's method at the discrete level.

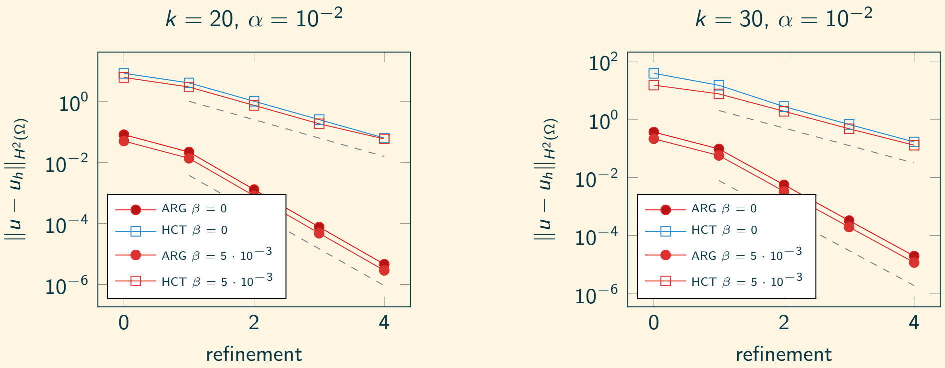
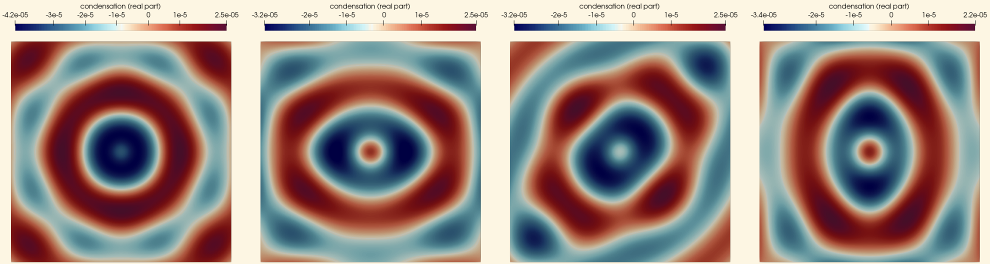


Figure: The convergence of the H^2-norm of the error for the Helmholtz–Korteweg equation for different values of k (top row) and the corresponding manufactured solution (bottom row).

ANISOTROPIC SPEED OF SOUND

We demonstrate the anisotropic speed of sound considering as right-hand side a symmetric Gaussian pulse in $(0,0)$, *impedance* BCs, $k = 40$, $\alpha = 10^{-2}$



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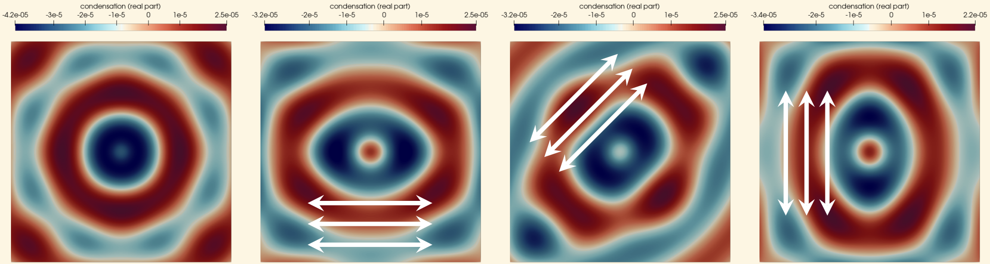

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TOTAL INTERNAL REFLECTION

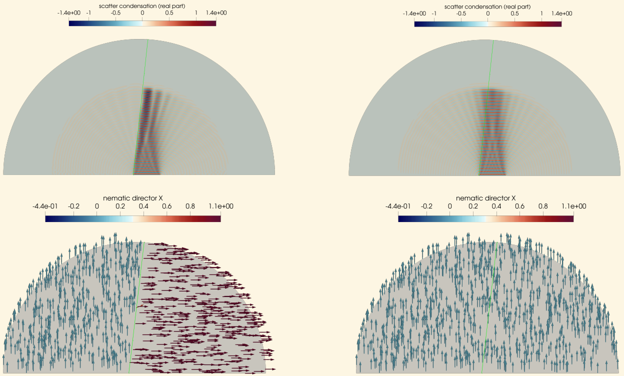
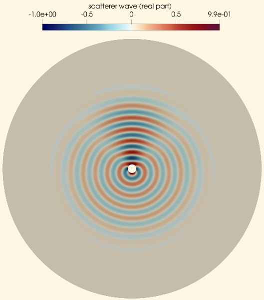


Figure: An acoustic reflection phenomenon in a nematic Korteweg fluid can be caused by a discontinuity in the nematic director field. We consider a Gaussian beam travelling upwards in a semicircular domain, with two different nematic director fields.

SCATTERING BY A CIRCULAR OBSTACLE



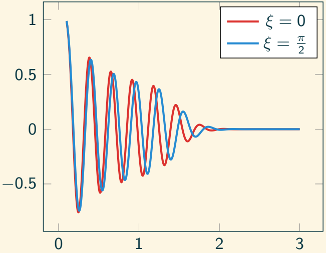


Figure: The scattered wave produced by a circular obstacle in a nematic Korteweg fluid with $\alpha = 10^{-3}$ and $u_2 = 5 \cdot 10^{-4}$, has a greater amplitude when the incoming plane wave is orthogonal to the nematic director field. ξ is the angle between the propagating direction of the plane wave $\boldsymbol{d}$ and $\boldsymbol{n}$. An adiabatic layer has been used to implement the Sommerfeld radiation condition on the outer bounday.

THANK YOU!

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https://doi.org/10.1103/PhysRevE.111.035413
https://doi.org/10.48550/arXiv.2502.17626

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