A Compressible Oseen–Frank Energy: Well-Posedness for Prescribed Pressure Fields

Umberto Zerbinati*, joint work with: J. Xia†, P. E. Farrell*

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EMS School: Mathematical Aspects of Fluid Flows, 2026

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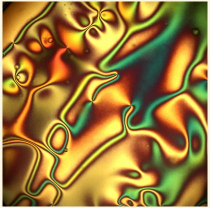
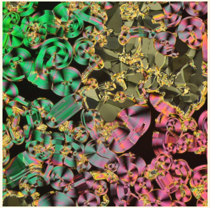
NEMATIC LIQUID CRYSTALS AND THEIR DEFECTS





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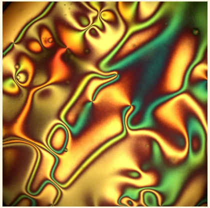
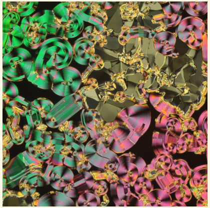
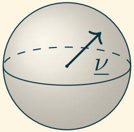
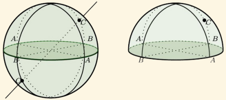






director on $\mathbb{S}^2$



$\mathbb{RP}^2$ (head–tail symmetry)

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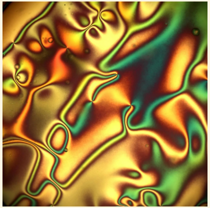

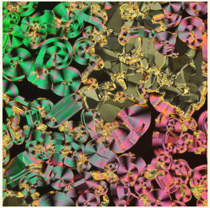

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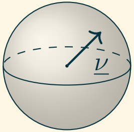

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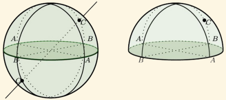

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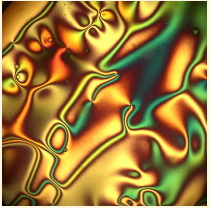

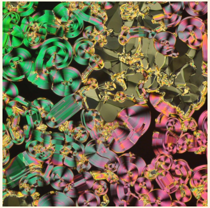

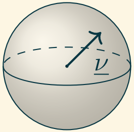

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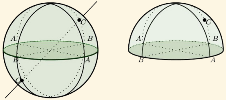

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Line defects have **infinite** energy $\Rightarrow$ excluded classically, yet seen in experiments.

A COMPRESSIBLE OSEEN–FRANK ENERGY

Oxford Mathematical Institute

Multiscale Model. Simul. (P. E. Farrell, G. Russo, U. Z.).

Kinetic theory (**Boltzmann–Curtiss**) yields an elastic energy **weighted by the thermodynamic pressure** $p = p(\rho, T)$:

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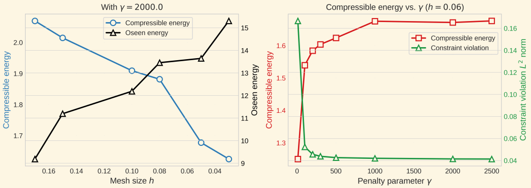
This talk: for a **prescribed** pressure p, minimise over the director, $\underline{\nu}_* = \operatorname{argmin}_{\underline{\nu} \in V} \mathcal{E}_{FR}[\underline{\nu}]$.

THE PAYOFF: LINE DEFECTS BECOME ADMISSIBLE

The **two-dimensional hedgehog** (a line defect)

$$\underline{\nu}(\underline{x}) = \frac{[x_1, x_2, 0]^\top}{\sqrt{x_1^2 + x_2^2}}$$

has **infinite** Oseen–Frank energy.



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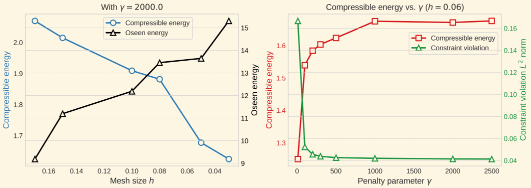

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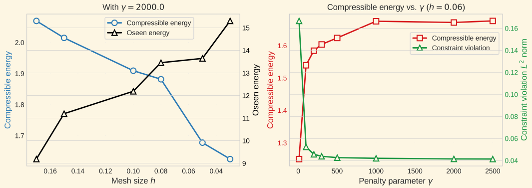

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The pressure **tames the singularity**: we must look beyond $W^{1,2}(\Omega, \mathbb{S}^2)$.

WEIGHTED SOBOLEV FRAMEWORK AND WELL-POSEDNESS

Trans. Amer. Math. Soc. (B. Muckenhoupt), *Weighted Sobolev Spaces* (A. Kufner), *Numer. Math.* (R. H. Nochetto, E. Otárola, A. J. Salgado).

The pressure is a **Muckenhoupt A_2 weight**. The right spaces are

$$V(\Omega) = W^{1,2}_p(\Omega, \mathbb{S}^2), \qquad Q(\Omega) = A_2(\Omega) \cap \mathcal{W}^{1,\infty}(\Omega).$$

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Direct method: weighted Poincaré $\Rightarrow$ bounded minimising sequence; compact embedding + weak lower semicontinuity $\Rightarrow$ minimizer. **Uniqueness fails** in general.

A WEIGHTED TRACE THEOREM

Numer. Math. (R. H. Nochetto, E. Otárola, A. J. Salgado) — weighted embeddings.

Prescribing $\underline{\nu}_0$ and closing the direct method needs a **trace** in the weighted setting — absent from the classical theory.

Theorem (weighted trace, new)

Let Ω be precompact with C^1 boundary and $p \in A_2(\Omega) \cap \mathcal{W}^{1,\infty}(\Omega)$ obey the growth bound $r^4/R^4 \leq C\, p(B_r)/p(B_R)$. Then a unique bounded linear trace $\gamma : W^{1,2}_p(\Omega, \mathbb{R}^3) \to L^2_{p|_{\partial\Omega}}(\partial\Omega, \mathbb{R}^3)$ exists, extends $\underline{\nu} \mapsto \underline{\nu}|_{\partial\Omega}$, and is **compact**.

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- **Idea:** FTC along the normal + weighted Hölder/Young; the growth bound limits how fast p vanishes on $\partial\Omega$.
- Compactness of γ passes the boundary condition to the weak limit $\Rightarrow$ **existence with prescribed** $\underline{\nu}_0$.

NUMERICS: AN EQUILIBRIUM THAT IS NOT A MINIMIZER

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Scheme. P^1 elements; unit length by penalty $\frac{\gamma}{2}\int(|\underline{\nu}_h|^2-1)^2$; Newton with line search; **deflated continuation** for multiple branches (Firedrake/PETSc).

Smooth equilibria solve the Euler–Lagrange equation

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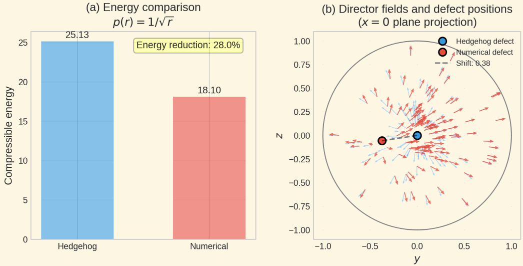

Left: energy of the hedgehog vs. the numerical minimizer. Right: the defect core shifts away from the origin under the non-constant pressure $p = 1/\sqrt{r}$.

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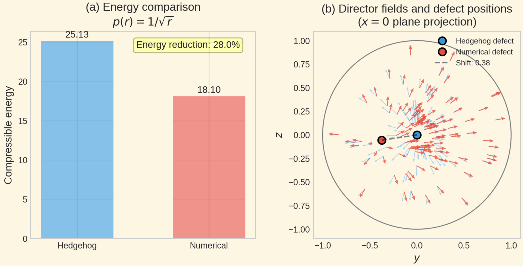
With $p=1/\sqrt{r}$ the **hedgehog** still *solves* this equation ($\mathcal{E}\approx 25.1$), but it is **not** the minimizer ($\mathcal{E}\approx 18.1$): the defect **migrates off-centre**.



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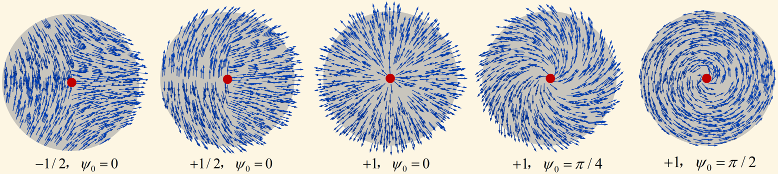
2D COMPUTATIONS ON THE DISC

Director fields on the unit disc (**deflated continuation**, P^1): finite-energy equilibria of **different topological degree**.



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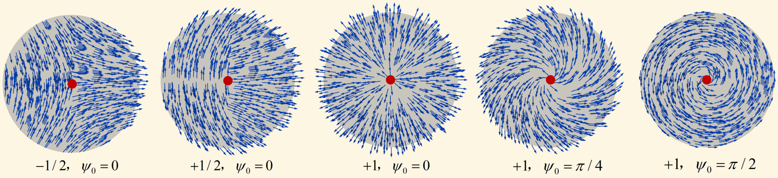
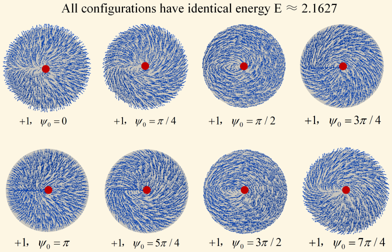




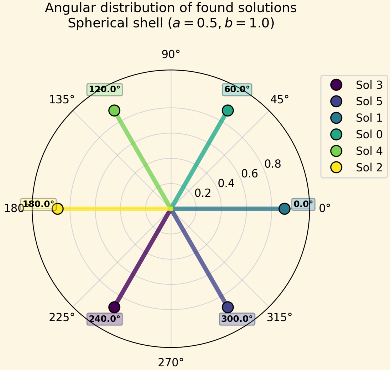
The degree-$+1$ defect comes in a **continuous family**: rotating the core angle ψ_0 leaves the energy unchanged, $\mathcal{E} \approx 2.1627$.

Uniqueness already fails in 2D.

NICE NUMERICS: A FAMILY OF MINIMIZERS ON A SHELL

On a shell $\Omega = \{a \leq |\underline{x}| \leq b\}$ with anti-parallel data $\underline{\nu} = \pm \underline{e}_z$ and $p = 1$:

- any horizontal axis $\underline{u} \in \mathbb{S}^1$ gives a minimizer $\underline{\nu} = [\underline{u} \sin\Theta(r), \cos\Theta(r)]^\top$ — a **continuous** $\mathbb{S}^1$ **family** of **equal energy**;



Azimuthal angles of six numerical minimizers at 60° spacing ($\gamma = 10^3$, $h = 0.05$, P^1).

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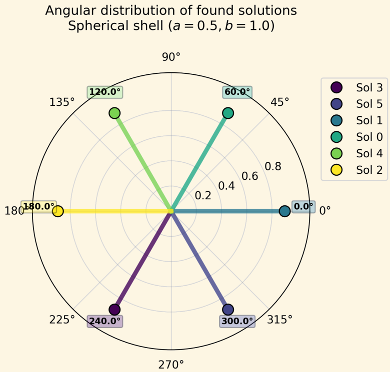

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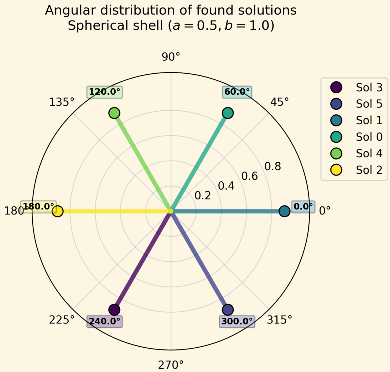
A concrete witness that **uniqueness fails**.



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BEYOND PRESCRIBED PRESSURE: THE STOKES COUPLING

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In a steady Stokes solvent, $-\nabla p + \mu \Delta \underline{u} = \underline{0}$, $\nabla \cdot \underline{u} = 0$ force the pressure to be **harmonic**, $\Delta p = 0$ — the minimiser of a Dirichlet energy. This suggests the **coupled** functional

$$\mathcal{E}[\underline{\nu}, p] = \int_\Omega \frac{1}{2}\, p \, |\nabla \underline{\nu}|^2 + \frac{1}{2}\, \kappa \, |\nabla p|^2 \, \mathrm{d}\underline{x}.$$

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- Setting $p = s^2$ gives $\int_\Omega \frac{1}{2}\, s^2 |\nabla \underline{\nu}|^2 + 2\kappa\, s^2 |\nabla s|^2 \, \mathrm{d}\underline{x}$.

BEYOND PRESCRIBED PRESSURE: THE STOKES COUPLING

Multiscale Model. Simul. (P. E. Farrell, G. Russo, U. Z.); *Manuscripta Math.* (L. Ambrosio).

In a steady Stokes solvent, $-\nabla p + \mu\Delta \underline{u} = \underline{0}$, $\nabla \cdot \underline{u} = 0$ force the pressure to be **harmonic**, $\Delta p = 0$ — the minimiser of a Dirichlet energy. This suggests the **coupled** functional

$$\mathcal{E}[\underline{\nu}, p] = \int_\Omega \tfrac{1}{2}\, p\, |\nabla \underline{\nu}|^2 + \tfrac{1}{2}\, \kappa\, |\nabla p|^2 \, \mathrm{d}\underline{x}.$$

- Setting $p = s^2$ gives $\int_\Omega \frac{1}{2}\, s^2 |\nabla \underline{\nu}|^2 + 2\kappa\, s^2 |\nabla s|^2 \, \mathrm{d}\underline{x}$.

This is not Ambrosio's functional

$$\mathcal{E}_A[\underline{\nu}, s] = \int_\Omega s^2 |\nabla \underline{\nu}|^2 + \kappa |\nabla s|^2 + \Psi(s) \, \mathrm{d}\underline{x}.$$

Ours weights the s-gradient by s^2 and carries **no potential** Ψ.

CONCLUSIONS AND OUTLOOK

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What we have shown

- A **kinetically derived** pressure weight regularises the elastic energy at defects.
- A **Muckenhoupt weighted-Sobolev** setting gives **well-posedness** thanks to a weighted **trace theorem**.
- The model **captures line defects** unreachable by classical Oseen–Frank.
- **Deflated continuation** reveals multiplicity and defect migration.

Outlook

- Full **coupling** of $\underline{\nu}$ and p.
- Beyond the **one-constant** approximation.
- **Uniqueness** and regularity of minimizers.

Thank you!

THANK YOU!

A Compressible Oseen–Frank Energy:
Well-Posedness for Prescribed Pressure Fields

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