On the symmetry constraint and angular momentum conservation in high order mixed stress formulation

Pablo Brubeck*, Charles Parker, II*, Umberto Zerbinati*

$*$ *Mathematical Institute – University of Oxford*

ICOSAHOM, Montreal, 18th July 2025

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Oxford Mathematics

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CONTINUUM MECHANICS – BALANCE LAWS

The governing equations of continuum mechanics are the conservation are usually regarded as the conservation of mass, linear momentum, angular momentum.

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) &= 0, \\ \rho\Big(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\Big) - \nabla \cdot \underline{\underline{\sigma}} &= \rho \boldsymbol{f}, \\ \rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta}\Big) - \nabla \cdot \underline{\underline{\mu}} - \boldsymbol{\xi} &= \boldsymbol{\rho\tau}, \end{aligned}$$

where ρ is the density, $\boldsymbol{u}$ is the either the linear momentum, $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, $\boldsymbol{\eta}$ is the intrinsic angular momentum, $\boldsymbol{\xi}$ is the antisymmetric part of the Cauchy stress tensor, $\underline{\underline{\mu}}$ is the couple stress tensor, $\boldsymbol{f}$ is the body force, and $\boldsymbol{\tau}$ is the body torque.

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The continuum mechanics governing equations need to be completed by **constitutive relations**.

SYMMETRY OF THE STRESS TENSOR

The symmetry of the Cauchy stress tensor lead to a conservation law for the angular momentum, if the body torque and the couple stress tensor vanish, i.e.

$$\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta}\Big) - \nabla\cdot\underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho\tau} \qquad .$$

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Conservation of Angular Momentum

Under the assumption that $\underline{\underline{\mu}} \equiv 0$, and $\boldsymbol{\tau} \equiv 0$, the symmetry of the Cauchy stress tensor, i.e. $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$, implies the conservation of the angular momentum, i.e. $\dot{\boldsymbol{\eta}} = 0$.

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Stokes Flow

A typical constitutive equation for the incompressible flow is the **Stokes flow**, which is given by

$$\underline{\underline{\sigma}} = 2\nu\underline{\underline{\varepsilon}}(\boldsymbol{u}) - p\underline{\underline{I}},$$

where ν is the kinematic viscosity, $\underline{\underline{\varepsilon}}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ is the strain rate tensor, and p is the Lagrange multiplier enforcing the incompressibility condition $\operatorname{div}\boldsymbol{u} = 0$.

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The Stokes flow is a linear problem, and it can be written in weak form as follows:

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}), \qquad b(\boldsymbol{u},q) = 0,$$

where $a(\boldsymbol{u},\boldsymbol{v}) = 2\nu(\underline{\underline{\varepsilon}}(\boldsymbol{u}),\underline{\underline{\varepsilon}}(\boldsymbol{v}))_{L^2(\Omega)}$ is the bilinear form associated with the viscous term, while $b(\boldsymbol{v},p) = (\nabla\cdot\boldsymbol{v},p)_{L^2(\Omega)}$ is the bilinear form for the incompressibility condition, and $(\boldsymbol{f},\boldsymbol{v})$ is the linear form for the body force.

STRUCTURE PRESERVING DISCRETISATIONS

A structure preserving discretisation is a numerical method that preserves some of the properties of the continuous problem, such as conservation laws, symmetries, or invariants.

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Divergence-Free Discretisations

The typical examples are divergence-free discretisations of the **incompressible constitutive relations**, where the div $\boldsymbol{u}_h = 0$ constraint is satisfied in a strong sense. This is can be achieved choosing as Q_h a space such that $\nabla \cdot V_h \subset Q_h$. In this case, the divergence free constraint is satisfied in a strong sense, i.e.

$$b(\boldsymbol{u}_h, q_h) = (\nabla \cdot \boldsymbol{u}_h, q_h)_{L^2(\Omega)} = 0 \quad \forall q_h \in Q_h.$$

becomes $\|\nabla \cdot \boldsymbol{u}_h\|_{L^2(\Omega)} = 0$, if we choose $q_h = \nabla \cdot \boldsymbol{u}_h$.

PRESSURE ROBUSTNESS – NO FLOW PROBLEM

J. V. Linke *et al.*, On the divergence constraint in mixed finite element methods for incompressible flows, **SIREV**, 2017.

A typical example used to demonstrate the pressure robustness exhibited by the divergence-free discretisations is the **no flow problem**, i.e.

$$\boldsymbol{f} = \begin{pmatrix} 0 \\ Ra(1-y+3y^2) \end{pmatrix}, \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad p = Ra(y^3 - \frac{1}{2}y^2 + y - \frac{7}{12}).$$

We expect the velocity to be independent of the pressure in the context of a divergence-free discretisation, contrary to the case of a non-divergence-free discretisation, i.e.

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} \leq C \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_{H^1(\Omega)} + C(\|\nabla\cdot\boldsymbol{u}_h\|_{L^2(\Omega)})\|p-p_h\|_{L^2(\Omega)}.$$

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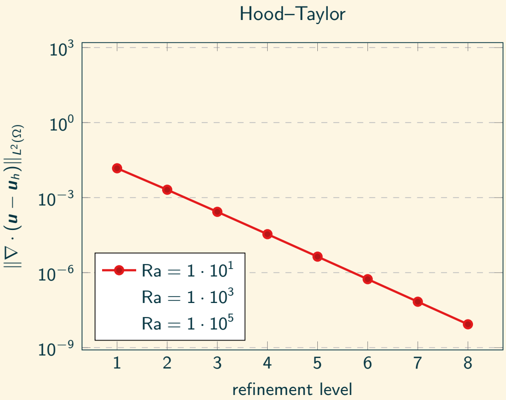

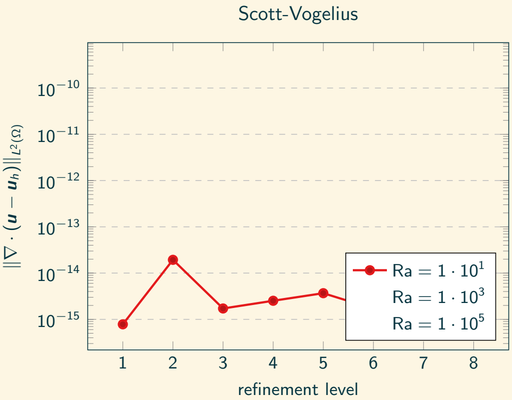

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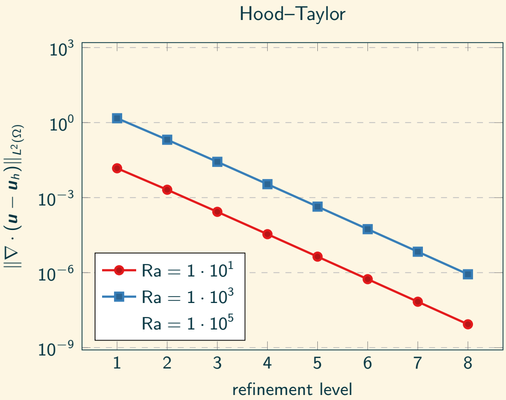

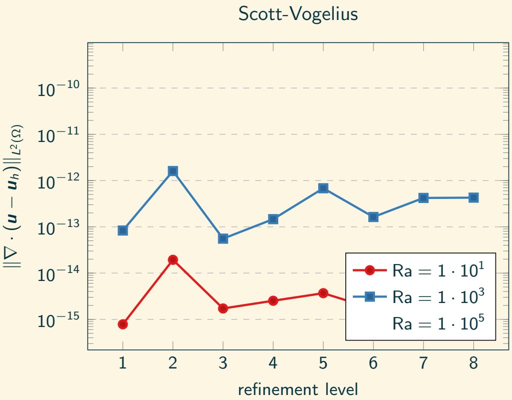

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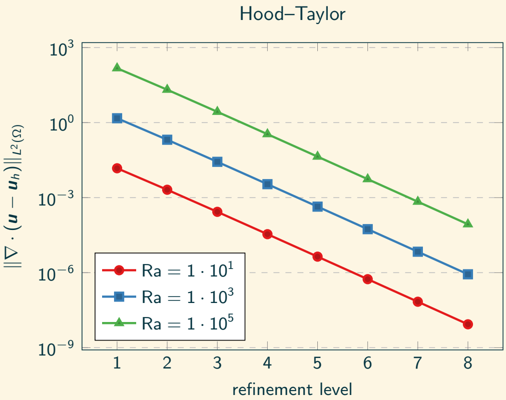

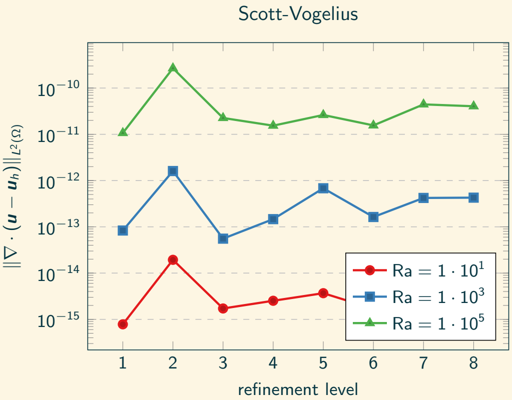

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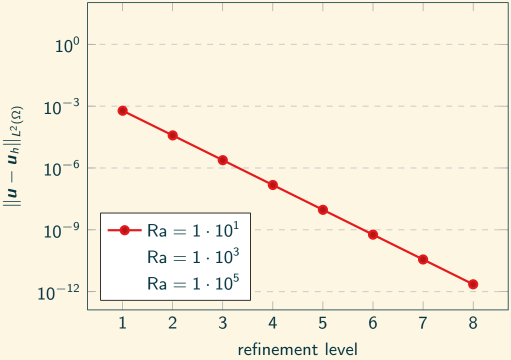

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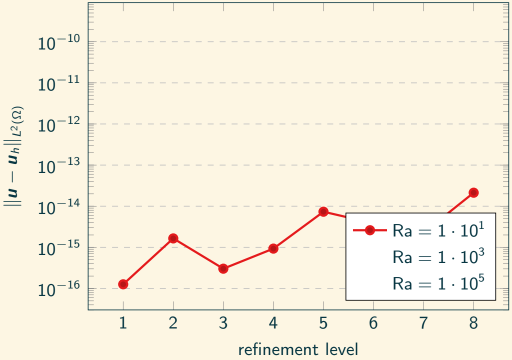

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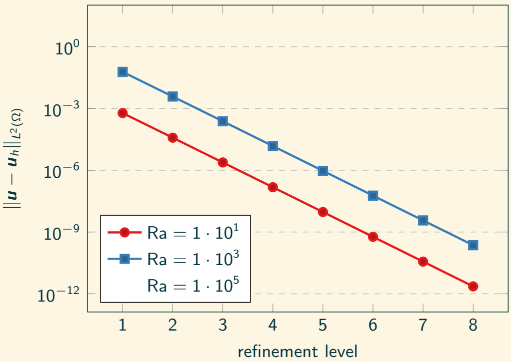

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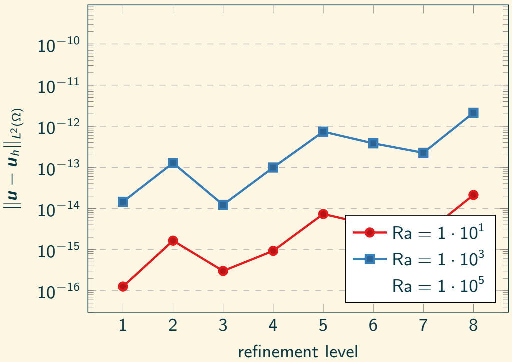

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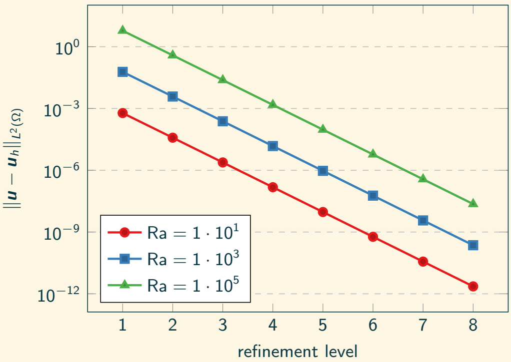

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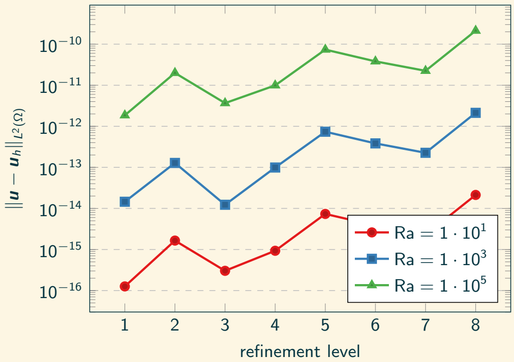

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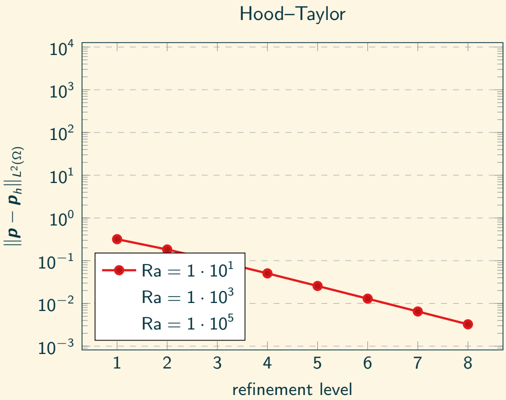

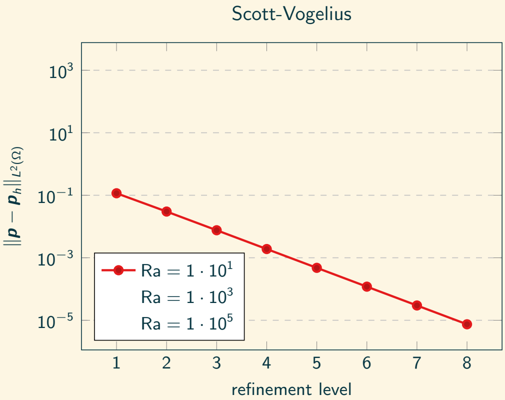

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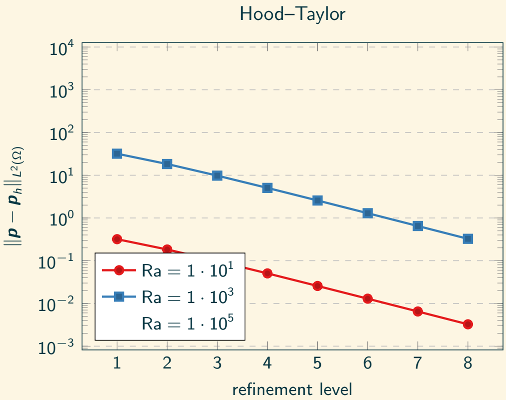

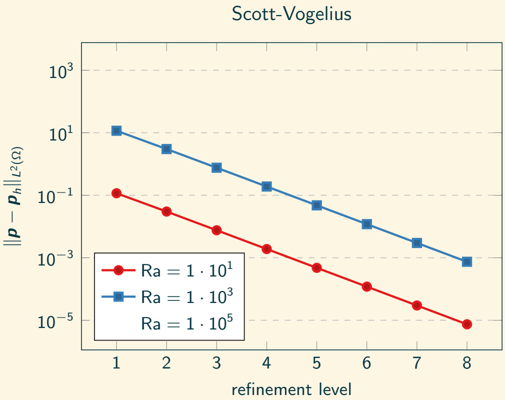

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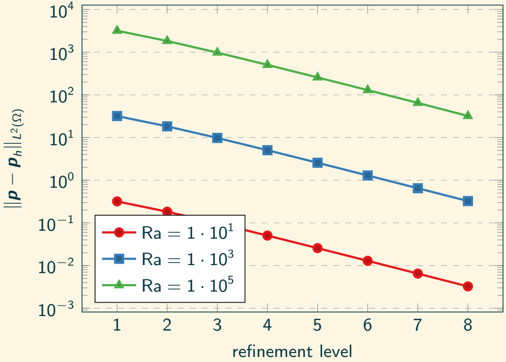

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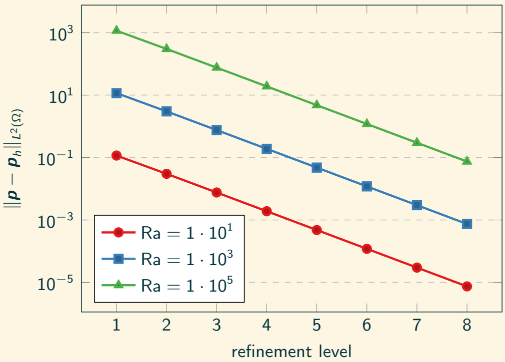

CONSERVATION OF MASS – NO FLOW PROBLEM

Considering the continuity equation, i.e.

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = \partial_t \rho + \rho \operatorname{div} \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \rho = \dot{\rho} + \rho \operatorname{div} \boldsymbol{u} = 0,$$

we obtain the evolution equation for the density,

$$\dot{\rho} = -\rho \operatorname{div} \boldsymbol{u}$$

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If the divergence-free condition is satisfied in a strong sense, i.e. $\operatorname{div} \boldsymbol{u}_h = 0$, then the density is constant, i.e. $\rho_h = \rho_0$, where ρ_0 is the initial density.

Let us begin considering a simpler yet related problem, namely the linear elasticity problem in stress formulation, i.e.

$$\begin{aligned} \operatorname{div} \underline{\underline{\sigma}} &= \boldsymbol{f}, \\ \underline{\underline{\sigma}} &= 2\mu \underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda\, \operatorname{tr}(\underline{\underline{\varepsilon}}(\boldsymbol{u}))\underline{\underline{\boldsymbol{I}}}, \end{aligned}$$

where $\boldsymbol{f}$ is once again the body force, μ is the shear modulus, λ is the first Lamé parameter.

ELASTICITY – STRESS FORMULATION

This problem can be written in weak form as follows:

$$\begin{aligned} a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) + b(\boldsymbol{u}, \underline{\underline{\tau}}) &= 0 \qquad \forall \underline{\underline{\tau}} \in \mathbb{S}_h \\ b(\underline{\underline{\sigma}}, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbb{V}_h \end{aligned}$$

$$a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) := \frac{1}{2\mu}(\underline{\underline{\sigma}}^D, \underline{\underline{\tau}}^D)_{L^2(\Omega)} + \frac{1}{4(\lambda+\mu)}(\mathrm{tr}(\underline{\underline{\sigma}}), \mathrm{tr}(\underline{\underline{\tau}}))_{L^2(\Omega)}, \qquad b(\boldsymbol{v}, \underline{\underline{\sigma}}) := (\mathrm{div}\, \underline{\underline{\sigma}}, \boldsymbol{v})_{L^2(\Omega)}$$

where the superscript D denotes the deviatoric part of the stress tensor, i.e. $\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - \frac{1}{2}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$.

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where the superscript D denotes the deviatoric part of the stress tensor, i.e. $\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - \frac{1}{2}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$.

To enforce the symmetry of the stress tensor, we can use introduce an additional constraint, i.e.

$$c(\underline{\underline{\sigma}}, \underline{\underline{\eta}}) := (\underline{\underline{\sigma}}, \underline{\underline{\eta}})_{L^2(\Omega)} = 0 \qquad \forall \underline{\underline{\eta}} \in \mathbb{AS}_h,$$

where $\mathbb{AS}_h$ is the space of antisymmetric tensors.

SYMMETRY CONSTRAINT – A PRIORI ERROR ESTIMATE

When reduced symmetry is imposed, the error estimate for the discrete scheme is fully coupled and take the form

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\underline{\underline{\eta}} - \underline{\underline{\eta}}_h\|_{L^2(\Omega)} \leq C\Bigg[\inf_{\tau_h \in \mathbb{S}_h} \|\underline{\underline{\sigma}} - \tau_h\|_{L^2(\Omega)} + \inf_{v_h \in \mathbb{V}_h} \|\boldsymbol{u} - v_h\|_{L^2(\Omega)} + \inf_{\eta_h \in \mathbb{AS}_h} \|\underline{\underline{\eta}} - \eta_h\|_{L^2(\Omega)}\Bigg].$$

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Strong Symmetry

If we impose the symmetry constraint, we obtain a decoupled error estimate of the form

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} \leq C \inf_{\tau_h \in \mathbb{S}_h} \|\underline{\underline{\sigma}} - \underline{\underline{\tau}}_h\|_{L^2(\Omega)}.$$

SYMMETRY CONSTRAINT – RIGID BODY MOTION

We begin from the most simple scenario, i.e. we try to induce a large component in the antisymmetric part of the stress tensor, via rigid body motion.

$$\boldsymbol{u} = C_{Bnd}\begin{pmatrix}-y\\ x\end{pmatrix}, \qquad \underline{\underline{\sigma}} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

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The exact solution are in the discrete spaces $[\mathbb{P}_1(\mathcal{T}_h)]^2$ and $[\mathbb{P}_0(\mathcal{T}_h)]^{2\times 2}$, hence $\underline{\underline{\eta}}$ can be approximated exactly by a **"low-order"** finite element approximation.

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The only elements in the kernel of the symmetric part of the gradient are the rigid body motions.

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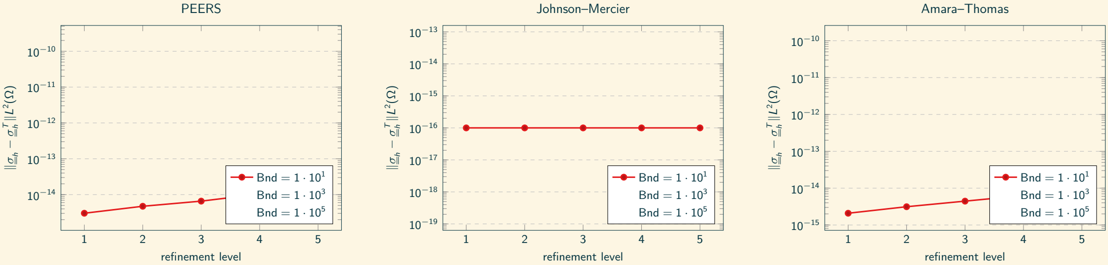

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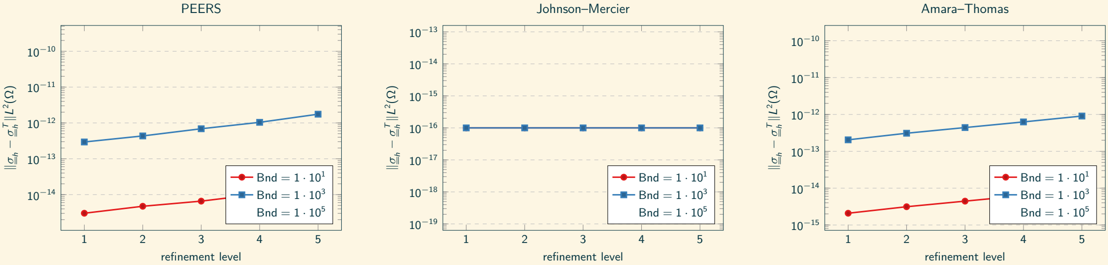

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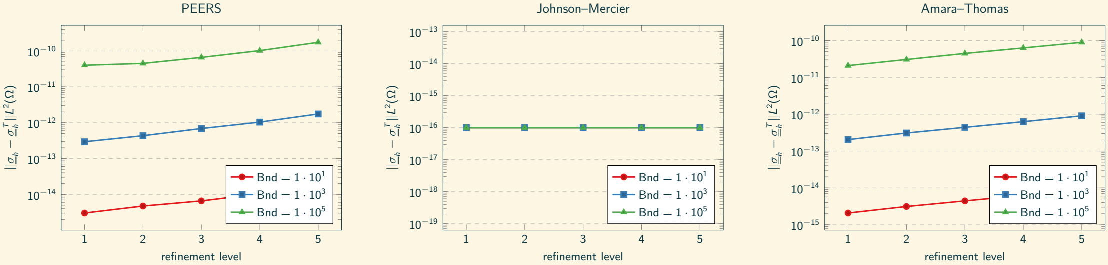

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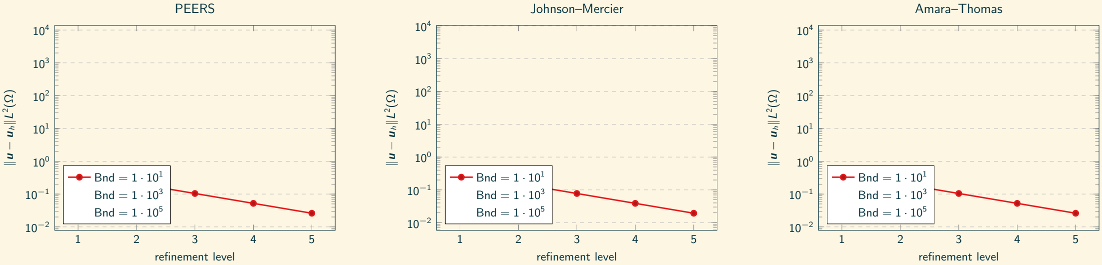

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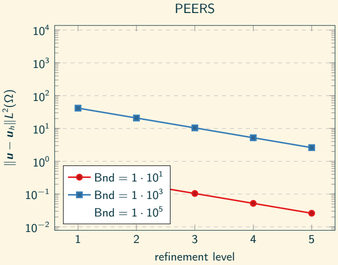

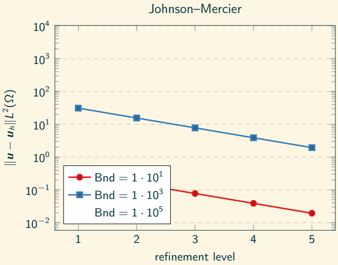

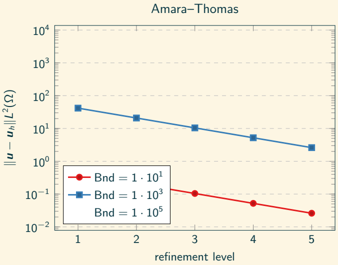

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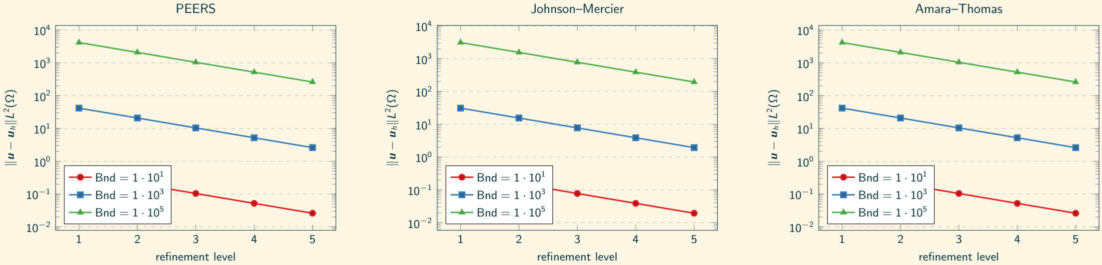

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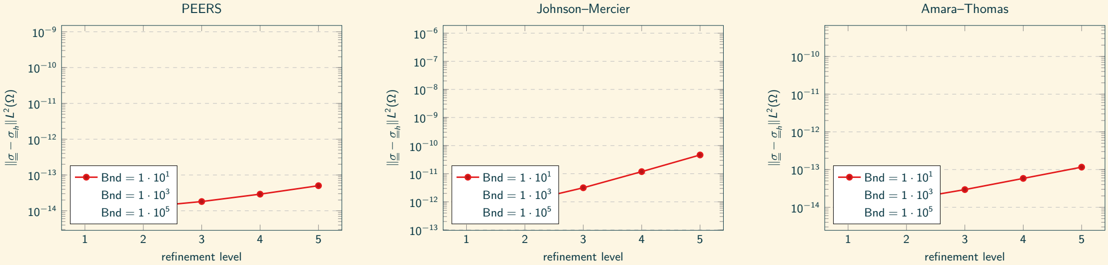

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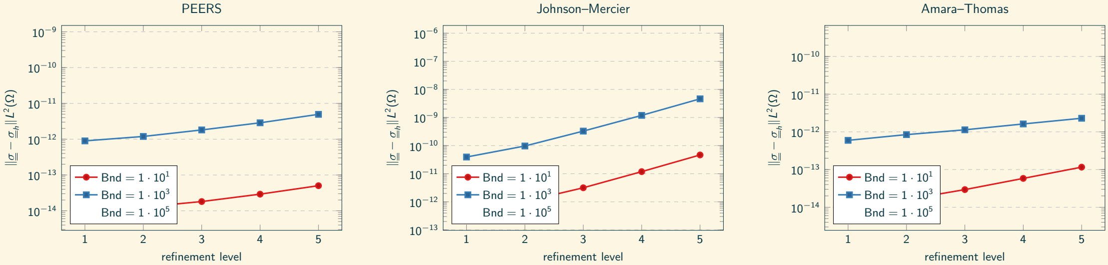

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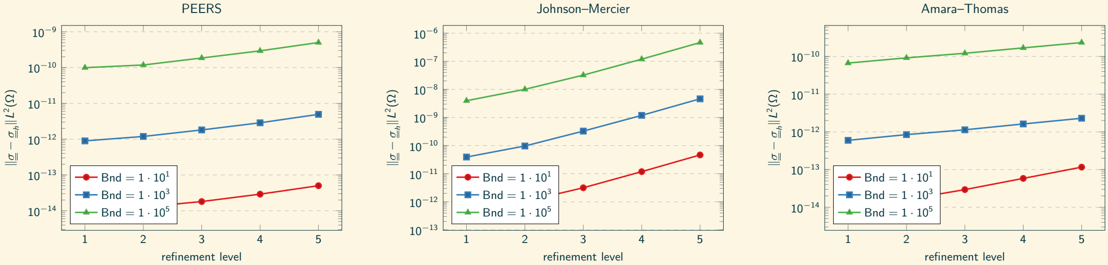

LIQUID CRYSTAL POLYMER NETWORKS – TRANSVERSE ANISOTROPY

T. .J. White, Photomechanical Effects in Liquid–Crystalline Polymer Networks and Elastomers, **J. Polymer Science**, 2017
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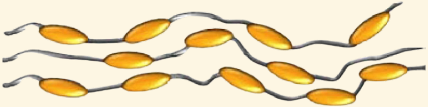
A liquid crystal polymer network (LCNs) is a material are **polymers** that exhibit a liquid crystalline phase, and are **crosslinked** to form a network structure, to obtain a material with unique mechanical properties. The most prominent example is **kevlar**.



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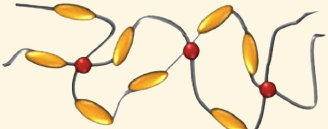

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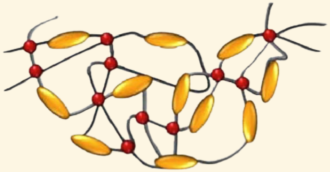

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Transversly Isotropic Material

LCNs exhibit a **transverse isotropy** in their mechanical properties, i.e. we can express the stress tensor as

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda(\nabla\cdot\boldsymbol{u})\underline{\underline{I}} + \boldsymbol{n}\otimes\boldsymbol{n}.$$

We here consider the following model problem, we pick

$$\boldsymbol{u} = -\frac{C_{Bnd}}{2\mu}\begin{pmatrix} \frac{1}{3}x^3 - \frac{2}{3}y^3 \\ x^2y + xy^2 + \frac{1}{3}y^3 + \frac{1}{3}x^3 \end{pmatrix}, \qquad \boldsymbol{n}(x,y) = C_{Bnd}^{\frac{1}{2}}\begin{pmatrix} x \\ x+y \end{pmatrix}.$$

There are also non rigid body motions in the kernel of the $\boldsymbol{u} \mapsto \underline{\underline{\sigma}}(\boldsymbol{u})$, thus the **strong** imposition of symmetry becomes important.

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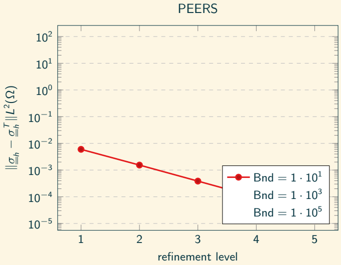

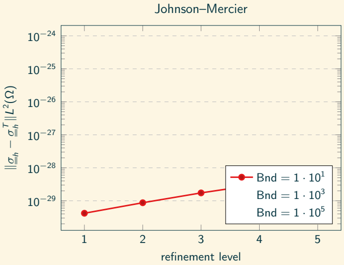

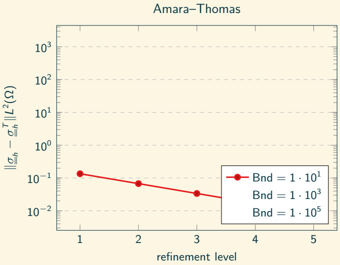

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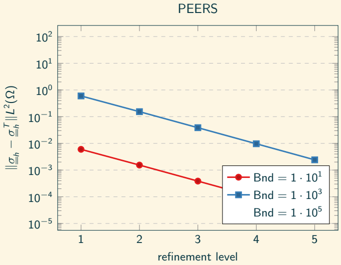

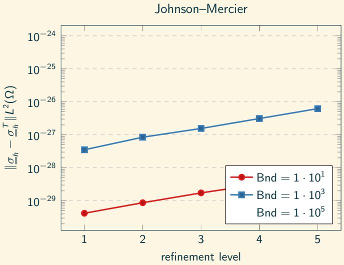

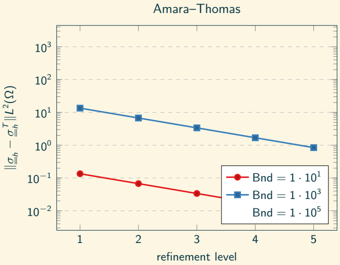

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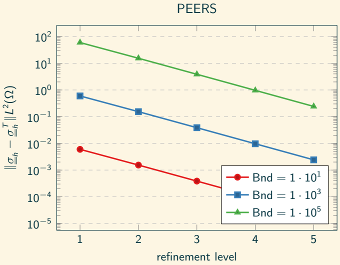

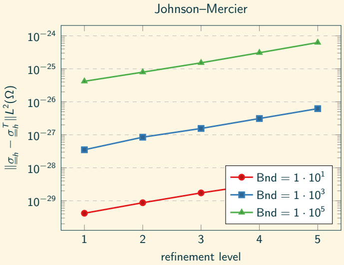

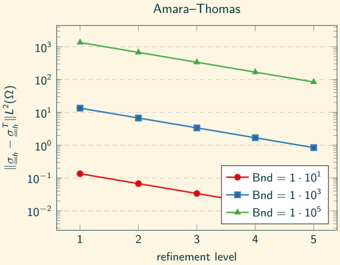

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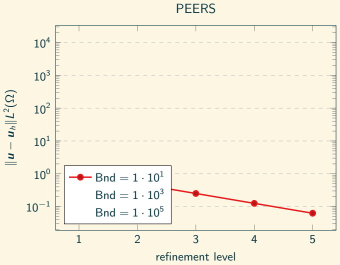

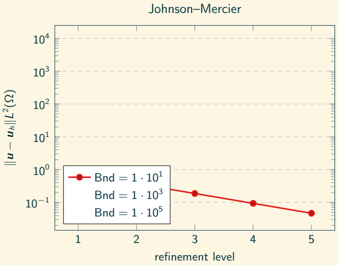

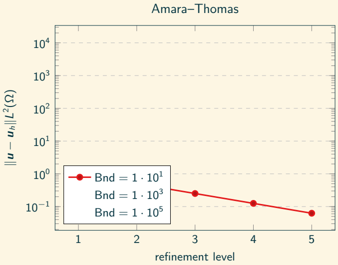

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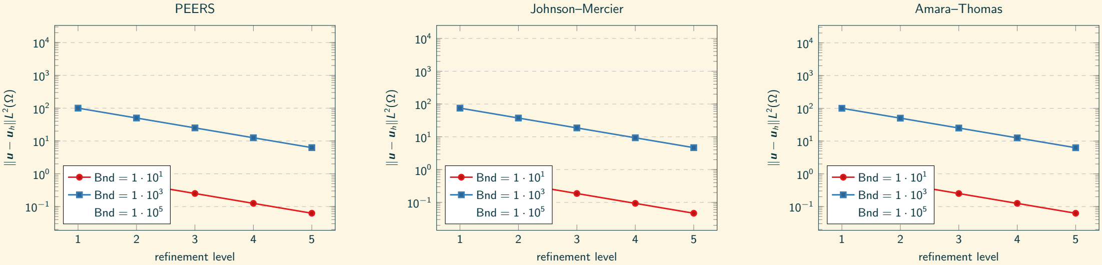

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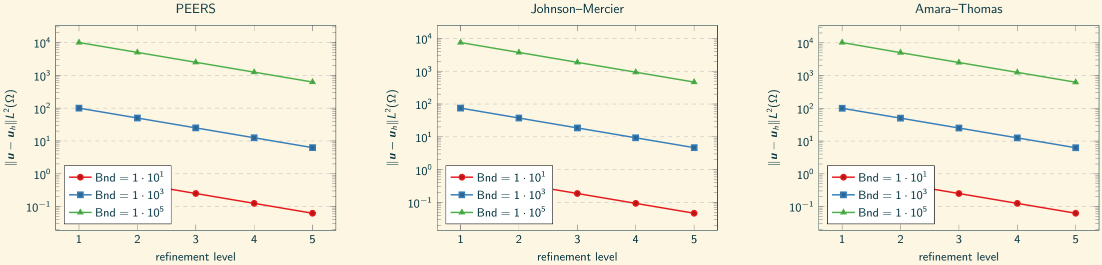

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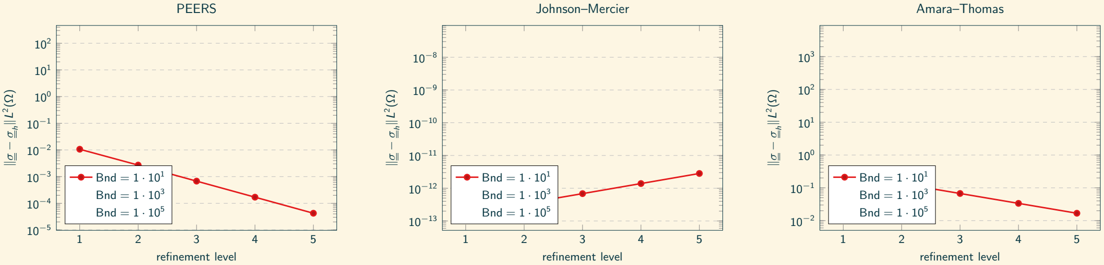

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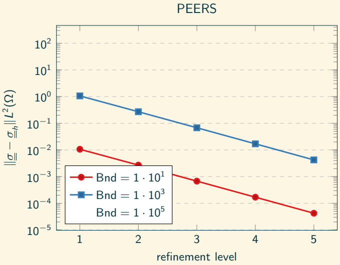

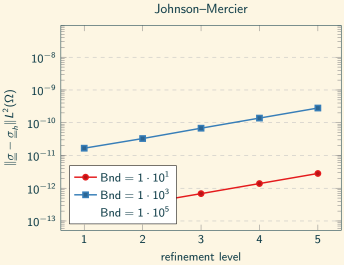

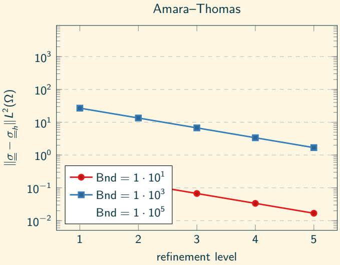

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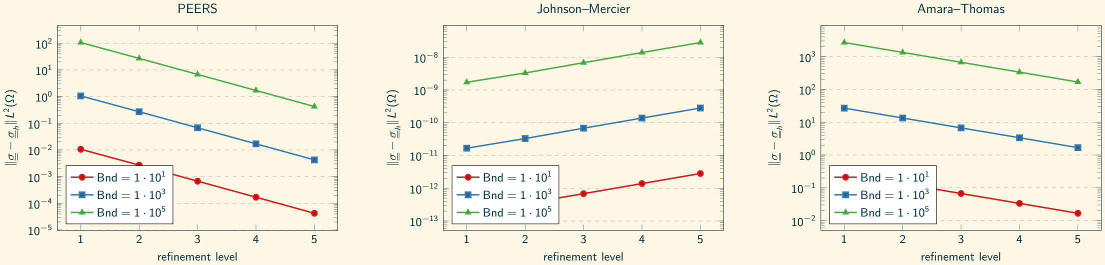

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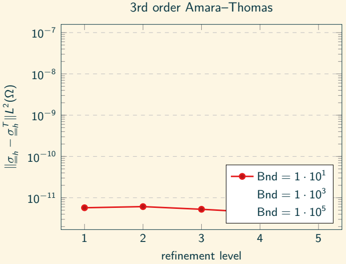

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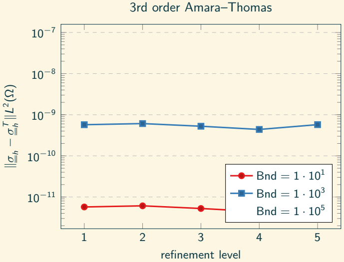

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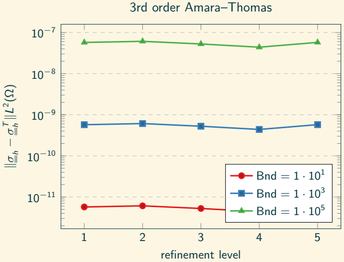

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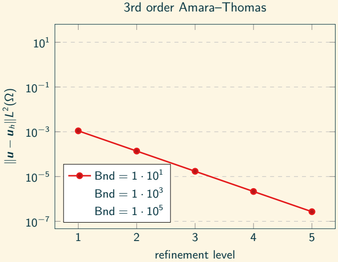

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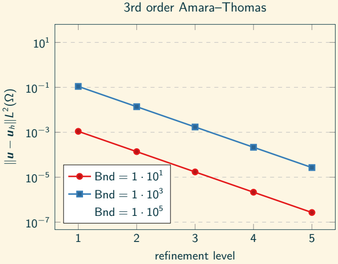

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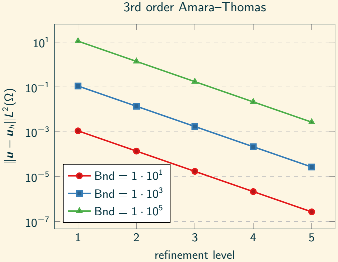

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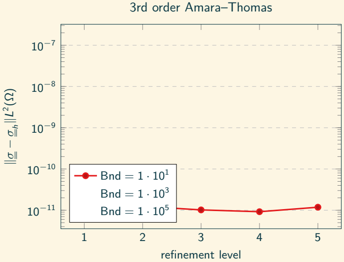

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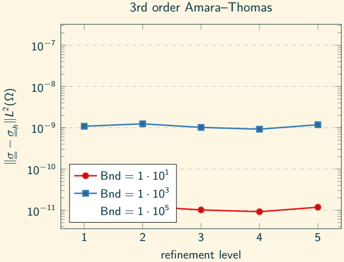

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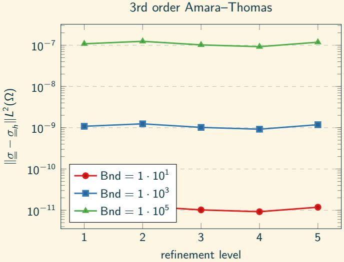

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$$\nabla \times \nabla \times \underline{\underline{\epsilon}} = 0.$$

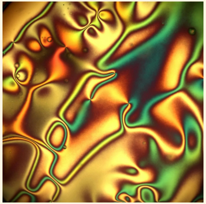
Saint–Venant compatibility condition for rank 1 tensors

Imposing the Saint–Venant compatibility condition when $\varepsilon(\boldsymbol{u}) = \boldsymbol{n} \otimes \boldsymbol{n}$, imposes the constraints that $\boldsymbol{n}$ is a **rigid body motion**.

LIQUID CRYSTALS FLUIDS – ERIOCKSEN STRESS TENSOR

J. L. Ericksen, Conservation laws for liquid crystals. Transactions of the Society of Rheology, 1961.

LIQUID CRYSTALS FLUIDS – ERIOCKSEN STRESS TENSOR



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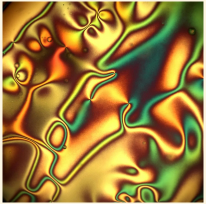
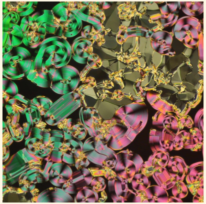

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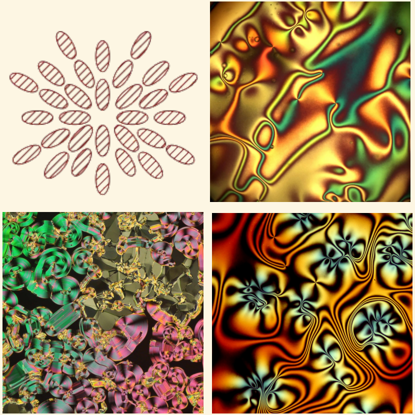

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Ericksen Stress Tensor

The Ericksen stress tensor is a symmetric rank 2 tensor, which is used to model the stress in liquid crystal materials, i.e.

$$\underline{\underline{\sigma}} = 2\nu \cdot \underline{\underline{\varepsilon}}(\boldsymbol{u}) + p\underline{\underline{I}} + K_F \cdot \nabla \boldsymbol{n}^T \nabla \boldsymbol{n}.$$

ERICKSEN TENSOR - PATCH TEST

We now design a patch test, for the **intrinsic** angular momentum, i.e.

$$\rho\left(\partial_t \boldsymbol{\eta} + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta}\right) - \nabla \cdot \underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho}\boldsymbol{\tau},$$

ERICKSEN TENSOR - PATCH TEST

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$$\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta}\Big) - \nabla \cdot \underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho\tau},$$

We pick a very silly couple stress tensor, i.e. $\underline{\underline{\mu}} = \nabla \boldsymbol{\eta}$, assume that η vanish at the boundary and have zero torque, i.e. $\boldsymbol{\tau} \equiv 0$.

PARAMETER ROBUSTNESS – ERICKSEN TENSOR

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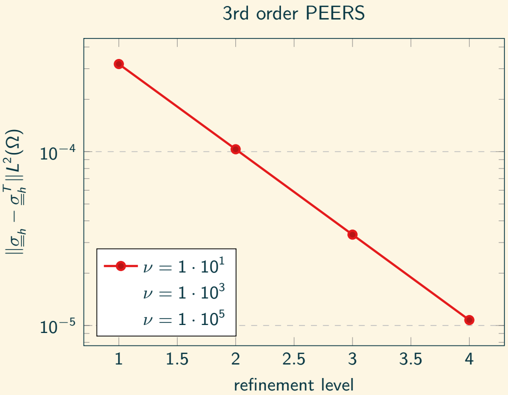

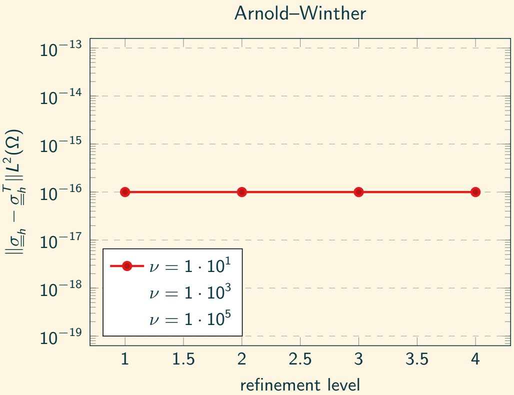

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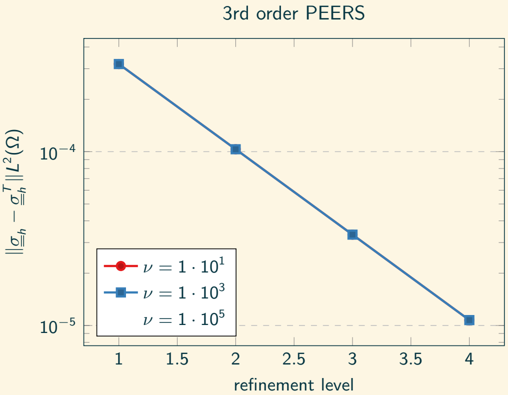

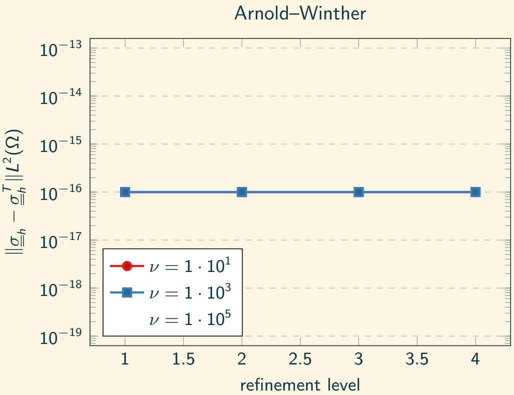

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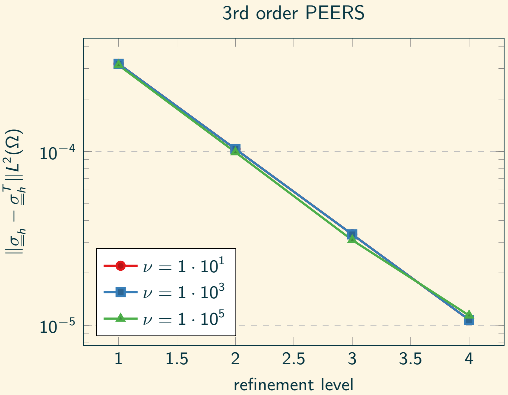

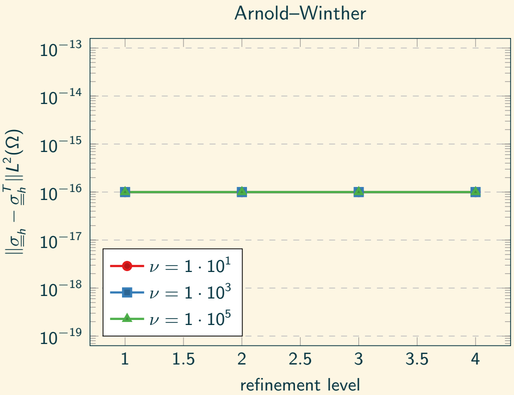

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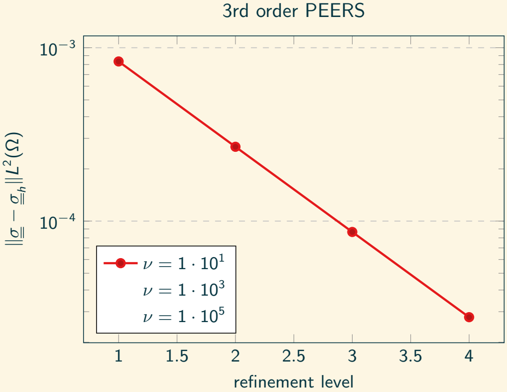

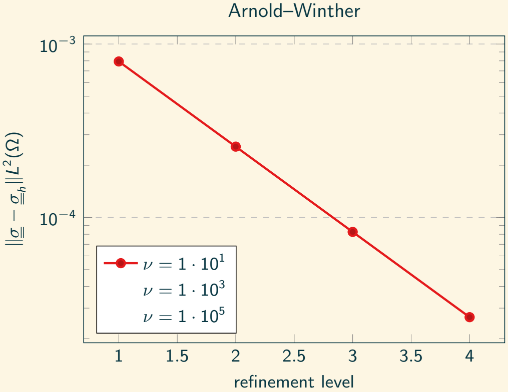

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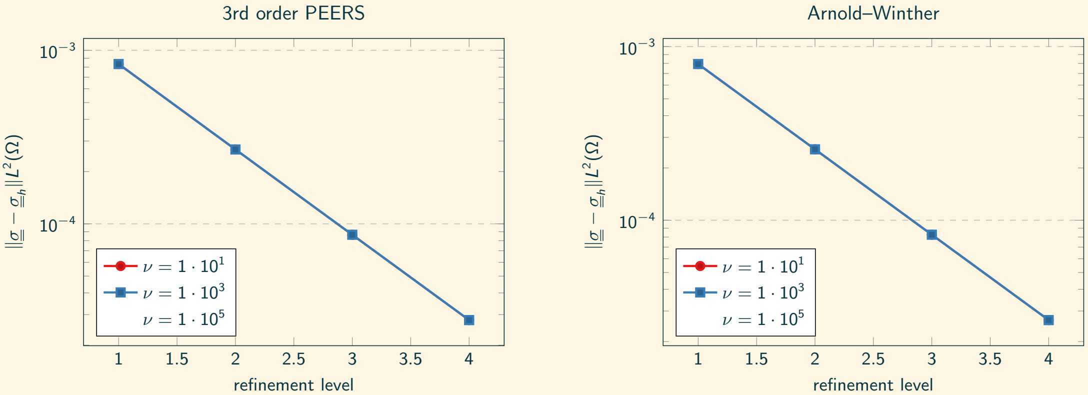

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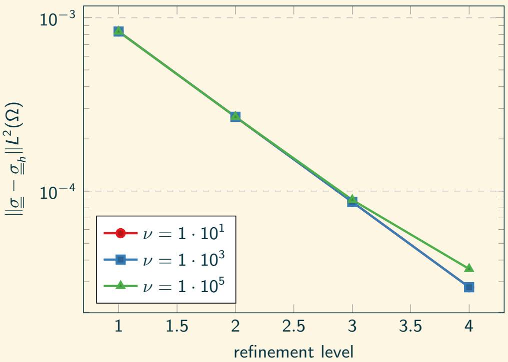
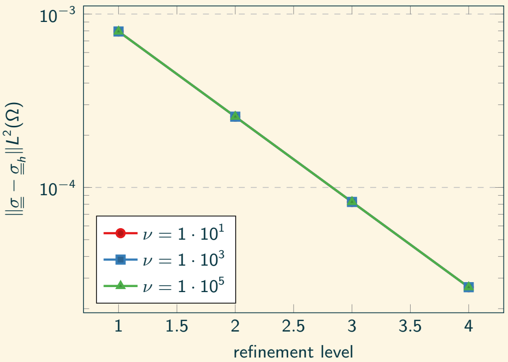
3rd order PEERS



Arnold–Winther



CONSERVATION OF ANGULAR MOMENTUM – ERICKSEN TENSOR

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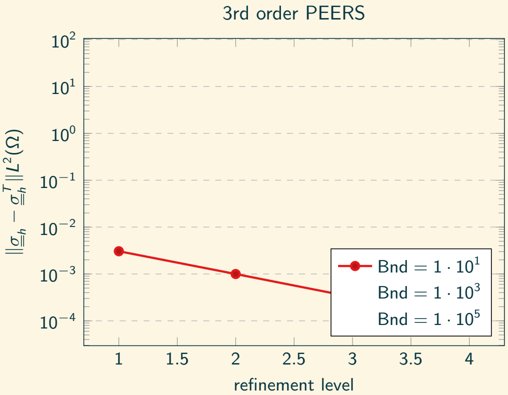

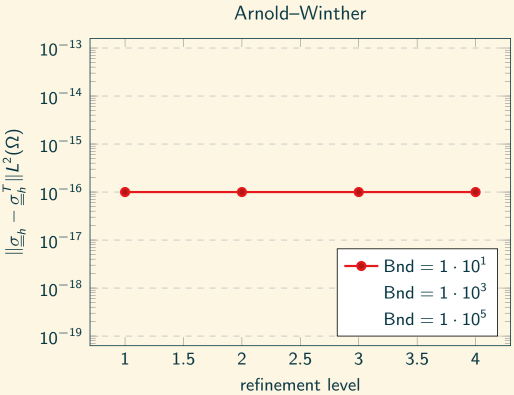

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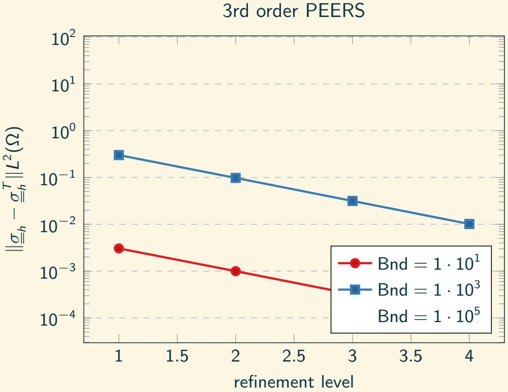

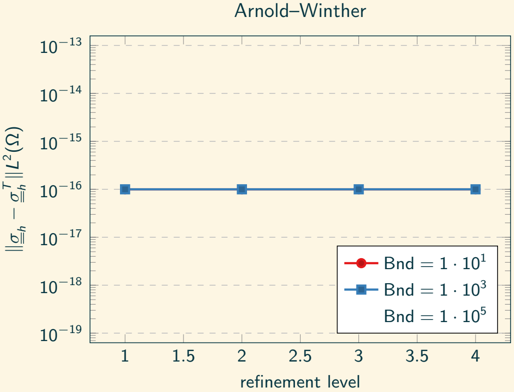

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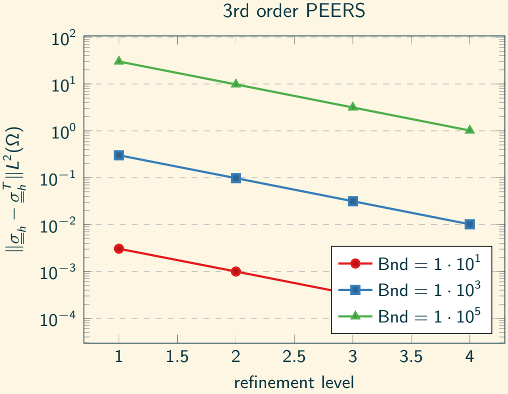

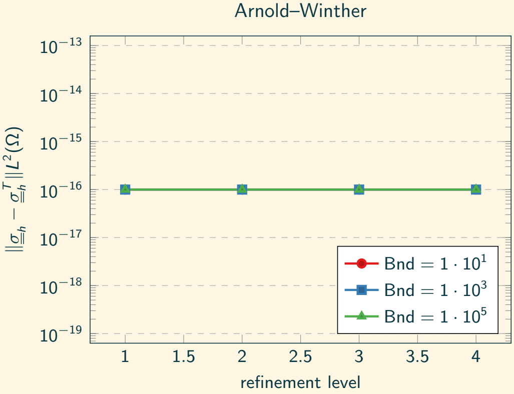

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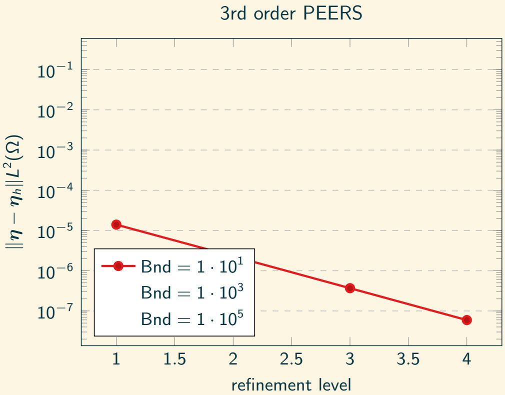

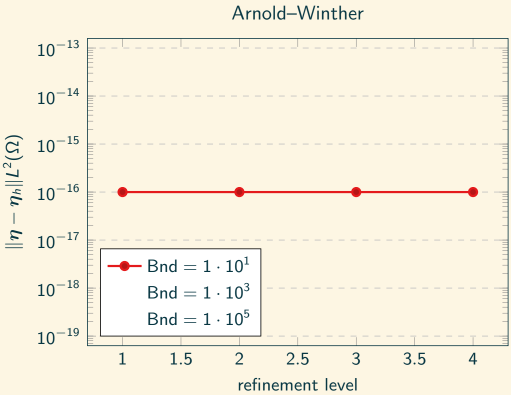

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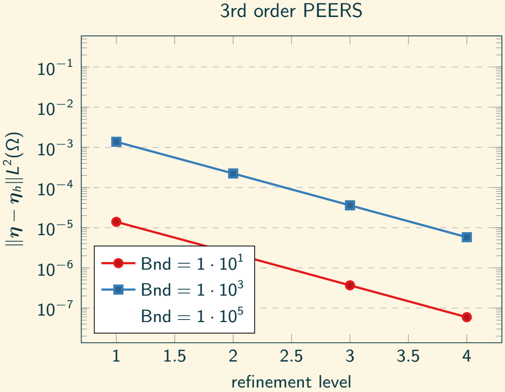

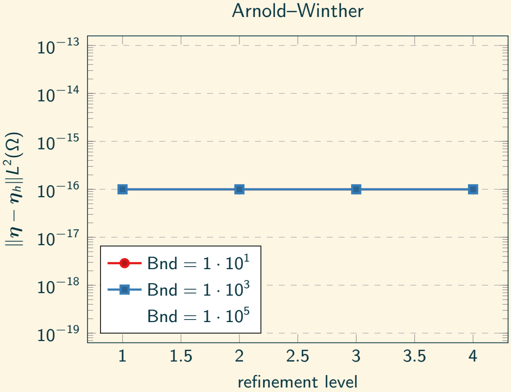

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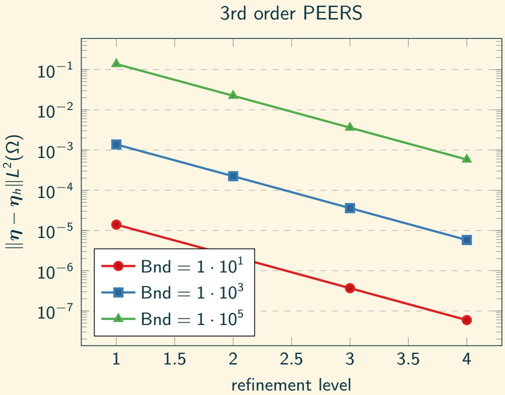

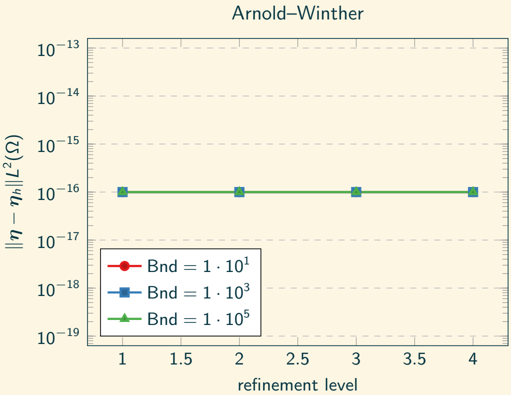

THANK YOU!

On the symmetry constraint and angular momentum conservation in high order mixed stress formulation

Pablo Brubeck*, Charles Parker, II*, Umberto Zerbinati*