On the symmetry constraint and angular momentum conservation in mixed stress formulation

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CONTINUUM MECHANICS – BALANCE LAWS

The governing equations of continuum mechanics are the conservation of mass, linear momentum, angular momentum.

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$
$$\rho\Big(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u}\Big) - \nabla\cdot \underline{\underline{\sigma}} = \rho \boldsymbol{f},$$
$$\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u}\cdot\nabla \boldsymbol{\eta}\Big) - \nabla\cdot \underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho}\boldsymbol{\tau},$$

where ρ is the density, $\boldsymbol{u}$ is the linear momentum per unit mass, $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, $\boldsymbol{\eta}$ is the intrinsic angular momentum, $\boldsymbol{\xi}$ is the antisymmetric part of the Cauchy stress tensor, $\underline{\underline{\mu}}$ is the couple stress tensor, $\boldsymbol{f}$ is the body force, and $\boldsymbol{\tau}$ is the body torque.

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The continuum mechanics governing equations need to be completed by **constitutive relations**.

SYMMETRY OF THE STRESS TENSOR

The symmetry of the Cauchy stress tensor leads to a conservation law for the angular momentum. If the body torque and the couple stress tensor vanish, we obtain

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Conservation of Angular Momentum

Under the assumption that $\underline{\underline{\mu}} \equiv 0$, and $\boldsymbol{\tau} \equiv 0$, the symmetry of the Cauchy stress tensor, i.e. $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$, implies the conservation of the angular momentum, i.e. $\dot{\boldsymbol{\eta}} = 0$.

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Stokes Flow

A typical constitutive equation for the incompressible flow is **Stokes's flow**, which is given by

$$\underline{\underline{\sigma}} = 2\nu\underline{\underline{\varepsilon}}(\boldsymbol{u}) - p\underline{\underline{I}},$$

where ν is the kinematic viscosity, $\underline{\underline{\varepsilon}}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ is the strain rate tensor, and p is the Lagrange multiplier enforcing the incompressibility condition $\operatorname{div}\boldsymbol{u} = 0$.

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The Stokes flow is a linear problem, and it can be written in weak form as follows:

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}), \qquad b(\boldsymbol{u},q) = 0,$$

where $a(\boldsymbol{u},\boldsymbol{v}) = 2\nu(\underline{\underline{\varepsilon}}(\boldsymbol{u}),\underline{\underline{\varepsilon}}(\boldsymbol{v}))_{L^2(\Omega)}$ is the bilinear form associated with the viscous term, while $b(\boldsymbol{v},p) = (\nabla\cdot\boldsymbol{v},p)_{L^2(\Omega)}$ is the bilinear form for the incompressibility condition, and $(\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)}$ is the linear form for the body force.

STRUCTURE PRESERVING DISCRETISATIONS

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Divergence-Free Discretisations

The typical examples are divergence-free discretisations of the **incompressible constitutive relations**, where the div $\boldsymbol{u}_h = 0$ constraint is satisfied in a strong sense. This is can be achieved choosing as Q_h a space such that $\nabla \cdot V_h \subset Q_h$. In this case, the divergence free constraint is satisfied in a strong sense, i.e.

$$b(\boldsymbol{u}_h, q_h) = (\nabla \cdot \boldsymbol{u}_h, q_h)_{L^2(\Omega)} = 0 \quad \forall q_h \in Q_h.$$

becomes $\|\nabla \cdot \boldsymbol{u}_h\|_{L^2(\Omega)} = 0$, if we choose $q_h = \nabla \cdot \boldsymbol{u}_h$.

PRESSURE ROBUSTNESS – NO FLOW PROBLEM

J. V. Linke *et al.*, On the divergence constraint in mixed finite element methods for incompressible flows, **SIREV**, 2017.

A typical example used to demonstrate the pressure robustness exhibited by the divergence-free discretisations is the **no flow problem**, i.e.

$$\boldsymbol{f} = \begin{pmatrix} 0 \\ Ra(1 - y + 3y^2) \end{pmatrix}, \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad p = Ra\left(y^3 - \frac{1}{2}y^2 + y - \frac{7}{12}\right).$$

We expect the velocity to be independent of the pressure in the context of a divergence-free discretisation, contrary to the case of a non-divergence-free discretisation, i.e.

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} \leq C \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{H^1(\Omega)} + C\big(\|\nabla \cdot \boldsymbol{u}_h\|_{L^2(\Omega)}\big)\|p - p_h\|_{L^2(\Omega)}.$$

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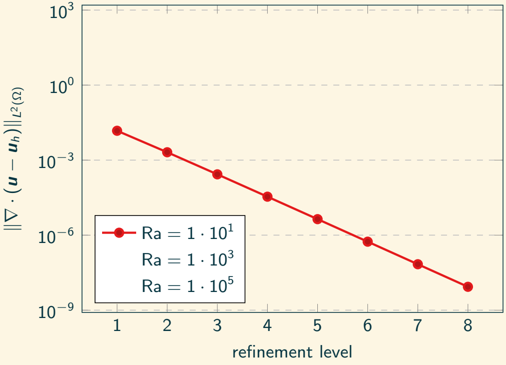
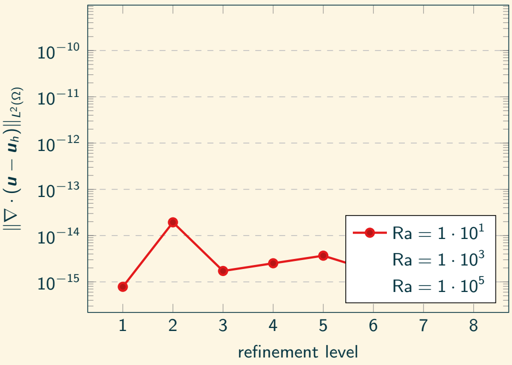
Hood–Taylor



Scott-Vogelius



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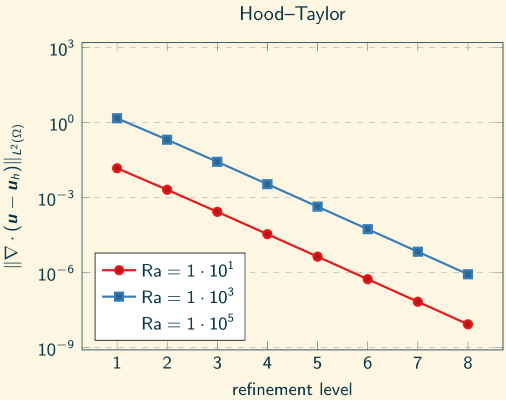

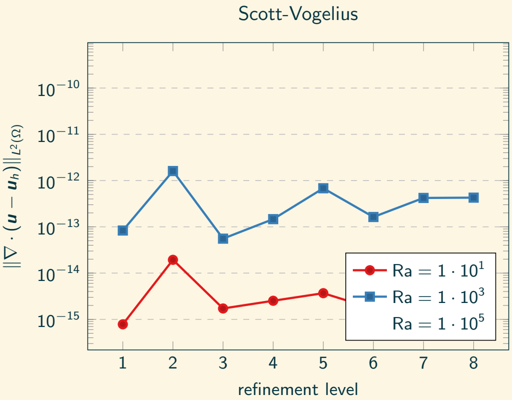

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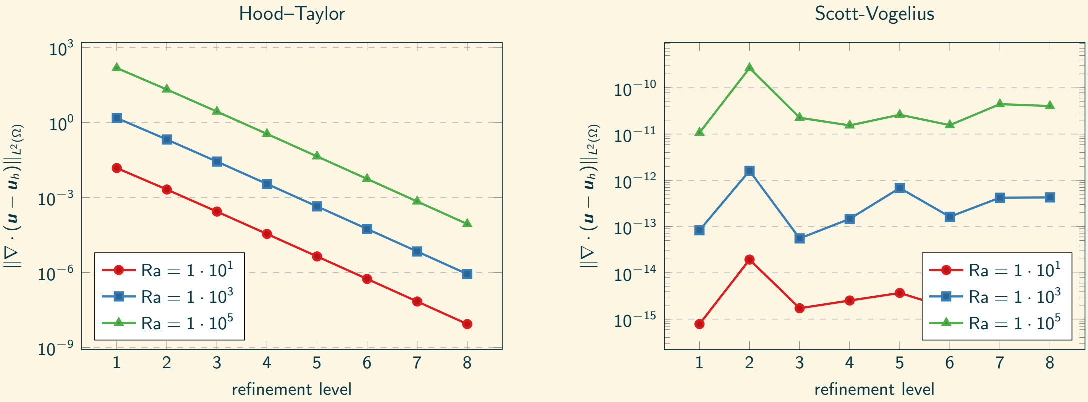

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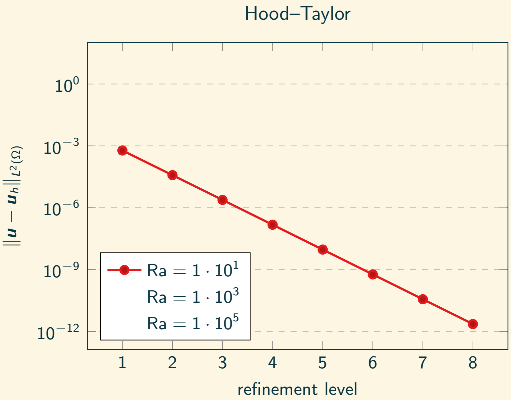

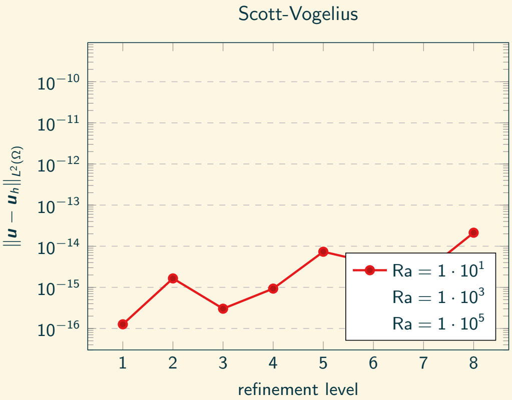

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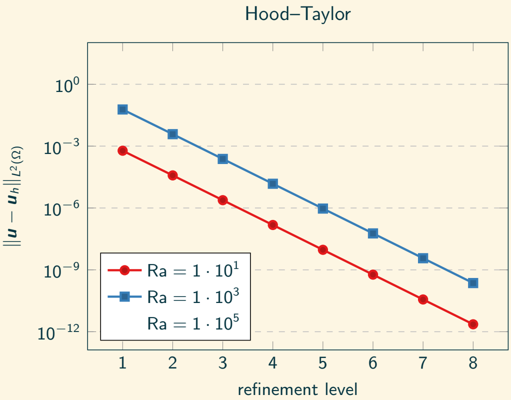

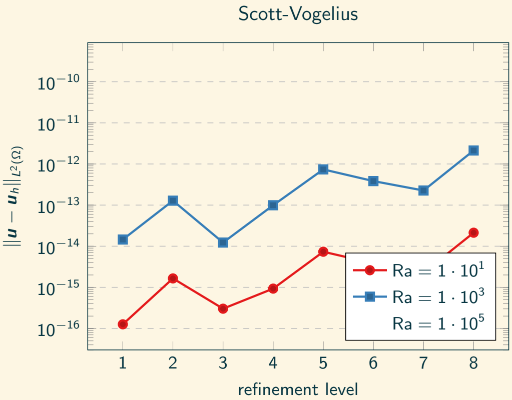

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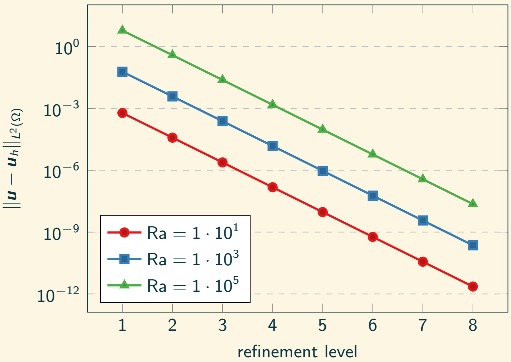

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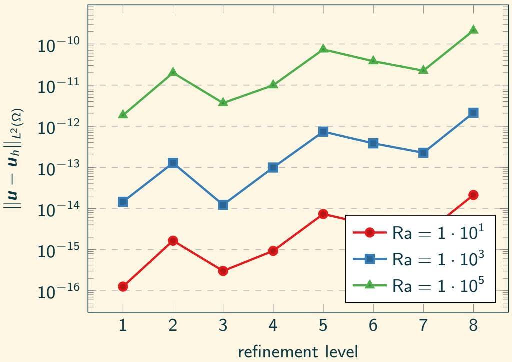

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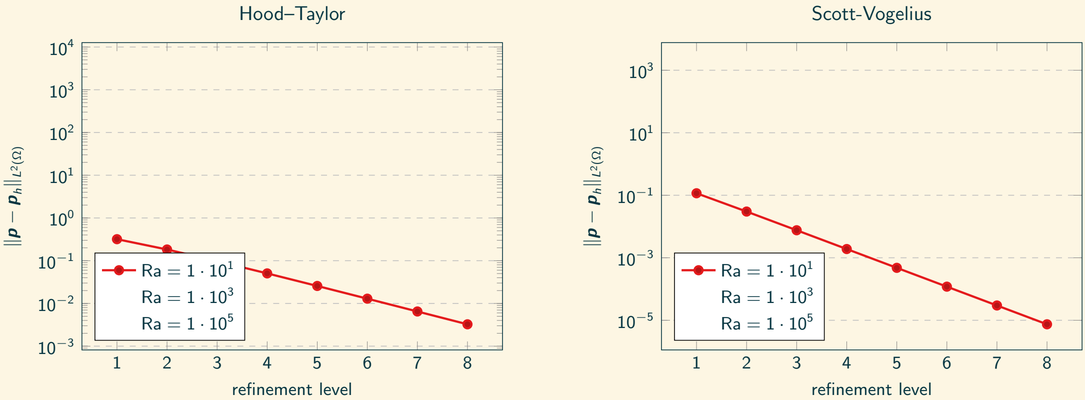

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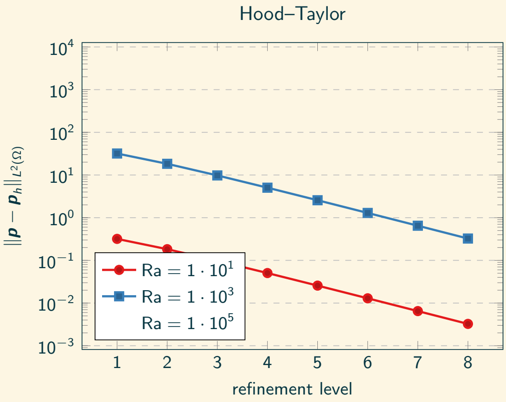

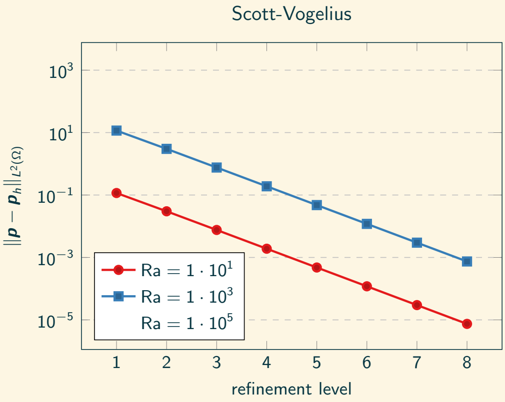

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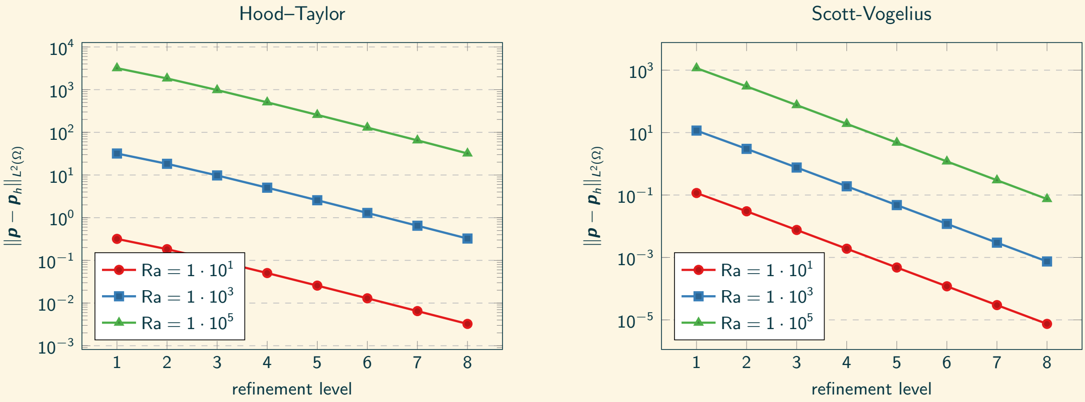

Let us begin considering a simpler yet related problem, namely the linear elasticity problem in stress formulation, i.e.

$$\operatorname{div} \underline{\underline{\sigma}} = \boldsymbol{f},$$
$$\underline{\underline{\sigma}} = 2\mu \underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda \operatorname{tr}(\underline{\underline{\varepsilon}}(\boldsymbol{u}))\underline{\underline{I}},$$

where $\boldsymbol{f}$ is once again the body force, $\boldsymbol{g}$ is the traction on the boundary $\partial\Omega$ and μ is the shear modulus, λ is the first Lamé parameter.

ELASTICITY – STRESS FORMULATION

This problem can be written in weak form as follows:

$$\begin{aligned} a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) + b(\boldsymbol{u}, \underline{\underline{\tau}}) &= \langle \underline{\underline{\tau}}\boldsymbol{n}, \boldsymbol{g} \rangle_{\partial\Omega} \qquad && \forall \underline{\underline{\tau}} \in \mathbb{S}_h \\ b(\boldsymbol{v}, \underline{\underline{\sigma}}) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in \mathbb{V}_h \end{aligned}$$

$$a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) := \frac{1}{2\mu}(\underline{\underline{\sigma}}^D, \underline{\underline{\tau}}^D)_{L^2(\Omega)} + \frac{1}{d(d\lambda + 2\mu)}(\mathrm{tr}(\underline{\underline{\sigma}}), \mathrm{tr}(\underline{\underline{\tau}}))_{L^2(\Omega)}, \qquad b(\boldsymbol{v}, \underline{\underline{\sigma}}) := (\mathrm{div}\, \underline{\underline{\sigma}}, \boldsymbol{v})_{L^2(\Omega)}$$

where the superscript D denotes the deviatoric part of a tensor, i.e. $\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - \frac{1}{d}\,\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$.

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To enforce the symmetry of the stress tensor, we introduce an additional Lagrange multiplier, i.e.

$$c(\underline{\underline{\sigma}},\underline{\underline{\eta}}) := (\underline{\underline{\sigma}},\underline{\underline{\eta}})_{L^2(\Omega)} = 0 \qquad \forall \underline{\underline{\eta}} \in \mathbb{AS}_h,$$

where $\mathbb{AS}_h$ is the space of antisymmetric tensors.

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PEERS

$$\mathbb{S}_h = \mathcal{RT}_k(\mathcal{T}_h)^{3r} + \mathrm{curl}(\mathcal{B}_{d+k}(\mathcal{T}_h)), \quad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega), \quad \mathbb{W}_h = \mathcal{P}_k(\mathcal{T}_h) \cap H^1(\Omega).$$

Arnold–Falk–Winther

$$\mathbb{S}_h = \mathcal{BDM}_k(\mathcal{T}_h)^{3r}, \quad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega), \quad \mathbb{W}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega).$$

Amara–Thomas

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When $k = 1$, notice that $\mathcal{BDFM}_1(\mathcal{T}_h)^{3r} = \mathcal{BDM}_1(\mathcal{T}_h)^{3r}$, thus this element is equivalent to the Arnold–Falk–Winther element of order 1.

Arnold–Winther

$$\mathbb{S}_h = \mathcal{AW}_k(\mathcal{T}_h), \quad \mathbb{V}_h = \mathcal{P}_{k-2}(\mathcal{T}_h) \cap L^2(\Omega).$$

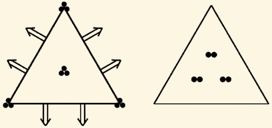

Figure: Arnold–Winther element of order $k = 3$ on a triangular mesh.

Johnson–Mercier

$$\mathbb{S}_h = \mathcal{JM}_k(\mathcal{T}_h), \quad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega).$$

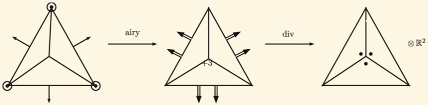

Figure: The complex leading to the Johnson–Mercier element of order $k = 1$ on a Alfeld mesh.

SYMMETRY CONSTRAINT – A PRIORI ERROR ESTIMATE

When reduced symmetry is imposed, the error estimate for the discrete scheme is fully coupled and takes the form

$$\begin{aligned}\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} + \mu\beta_h\left[\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\underline{\underline{\eta}} - \underline{\underline{\eta}}_h\|_{L^2(\Omega)}\right] \leq\; & C\beta_h^{-1}\inf_{\tau_h\in\mathbb{S}_h}\|\underline{\underline{\sigma}} - \tau_h\|_{L^2(\Omega)}\\ &+ C\mu\inf_{v_h\in\mathbb{V}_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Omega)}\\ &+ C\mu\inf_{\eta_h\in\mathbb{AS}_h}\|\underline{\underline{\eta}} - \eta_h\|_{L^2(\Omega)}.\end{aligned}$$

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$$+ C\mu\inf_{v_h\in\mathbb{V}_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Omega)}$$
$$+ C\mu\inf_{\eta_h\in\mathbb{AS}_h}\|\underline{\underline{\eta}} - \eta_h\|_{L^2(\Omega)}.$$

Strong Symmetry

If we impose the symmetry constraint and $\nabla\cdot\mathbb{S}_h = \mathbb{V}_h$, we obtain a decoupled error estimate of the form

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} \leq C\beta_h^{-1}\inf_{\tau_h\in\mathbb{S}_h}\|\underline{\underline{\sigma}} - \underline{\underline{\tau}}_h\|_{L^2(\Omega)}.$$

PATCH TEST – RIGID BODY MOTION

We begin from the most simple scenario, i.e. we try to induce a large component in the antisymmetric part of the stress tensor, via rigid body motion.

$$\boldsymbol{u} = C_{Bnd} \begin{pmatrix} -y \\ x \end{pmatrix}, \qquad \underline{\underline{\sigma}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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The exact solution are in the discrete spaces $[\mathbb{P}_1(\mathcal{T}_h)]^2$ and $[\mathbb{P}_0(\mathcal{T}_h)]^{2\times 2}$, hence $\underline{\underline{\eta}}$ can be approximated exactly by a **"low-order"** finite element approximation.

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The only elements in the kernel of the symmetric part of the gradient are the rigid body motions.

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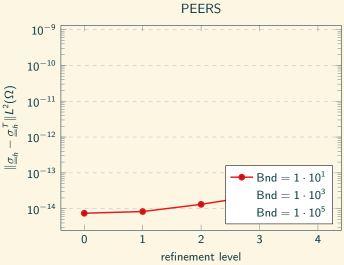

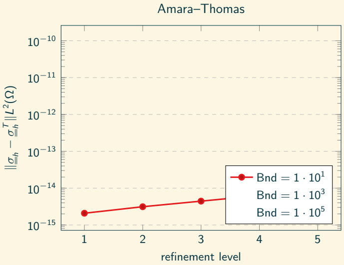

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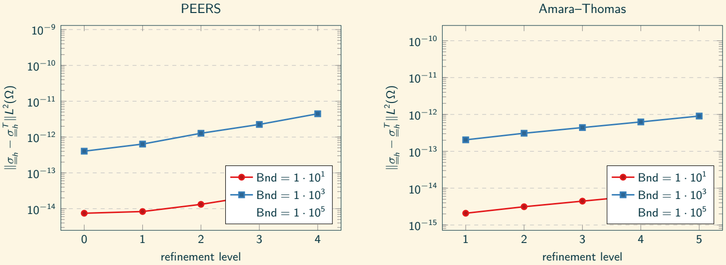

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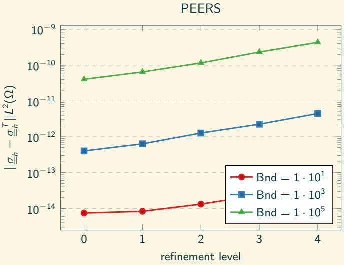

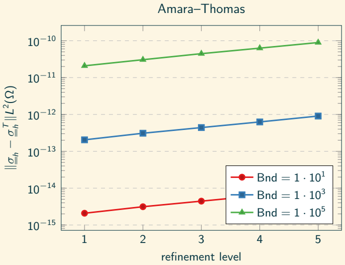

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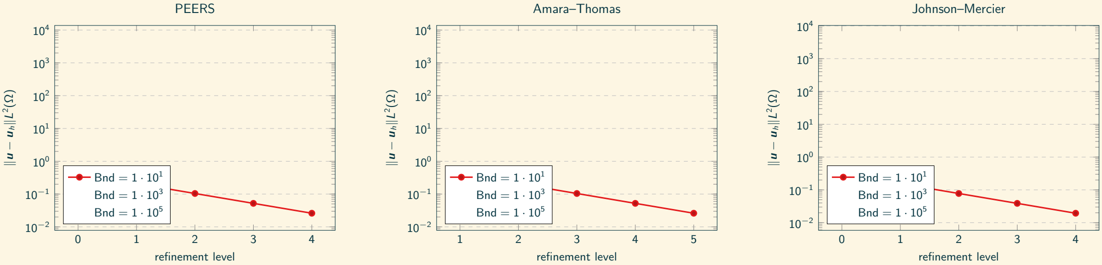

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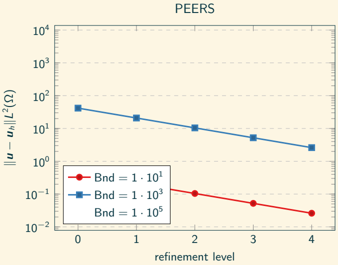

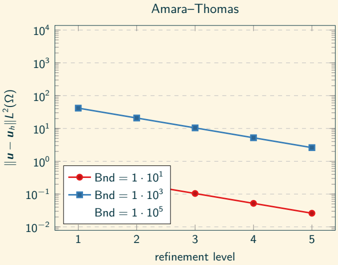

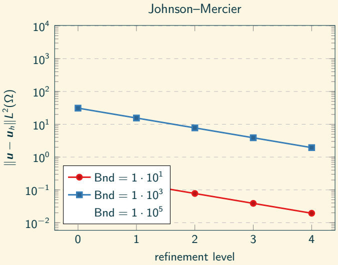

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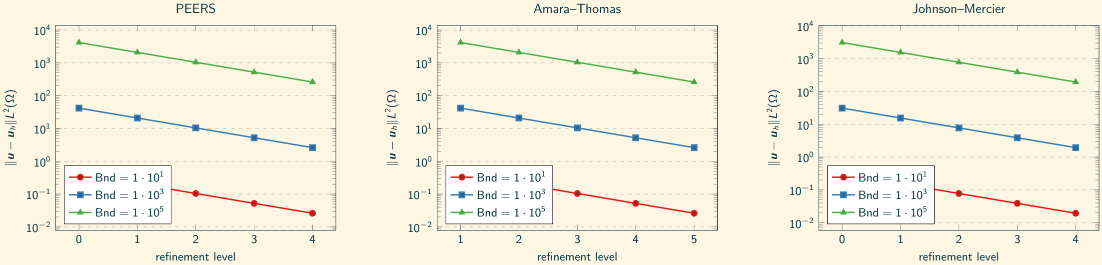

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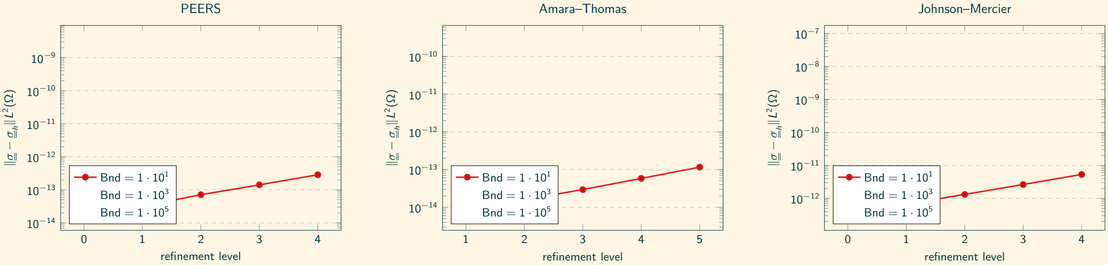

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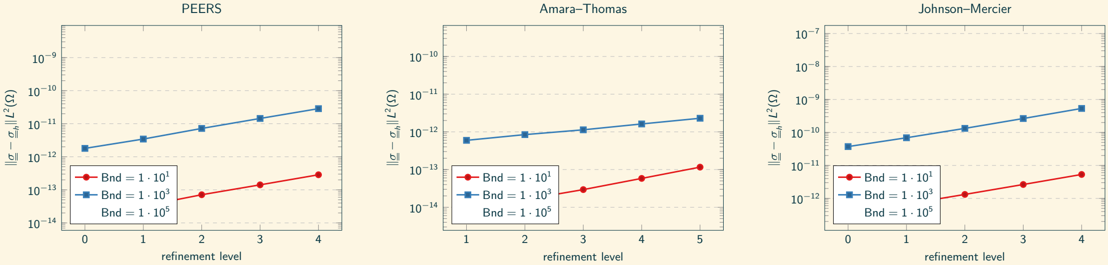

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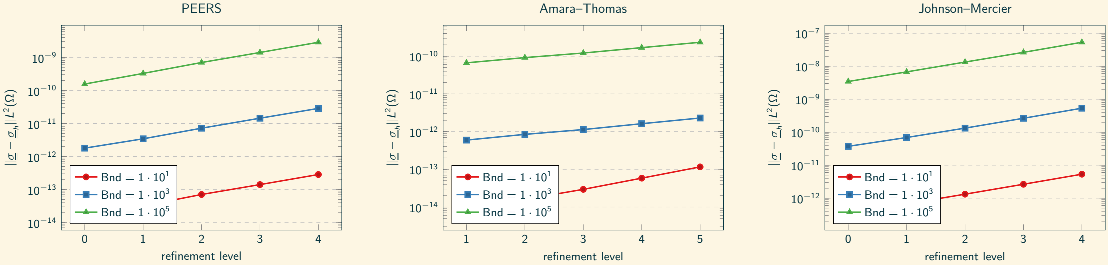

LIQUID CRYSTAL POLYMER NETWORKS – TRANSVERSE ANISOTROPY

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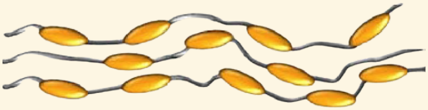
A liquid crystal polymer network (LCNs) is a material are **polymers** that exhibit a liquid crystalline phase, and are **crosslinked** to form a network structure, to obtain a material with unique mechanical properties. The most prominent example is **kevlar**.



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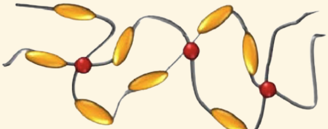

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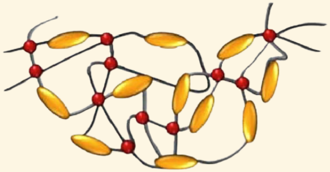

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Transversly Isotropic Material

LCNs exhibit a **transverse isotropy** in their mechanical properties, i.e. we can express the stress tensor as

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\underline{\underline{I}} + \boldsymbol{n} \otimes \boldsymbol{n}.$$

PATCH TEST – TRANSVERSE ANISOTROPY

We here consider the following model problem, we pick

$$\boldsymbol{u} = -\frac{C_{Bnd}}{2\mu}\begin{pmatrix} \frac{1}{3}x^3 - \frac{2}{3}y^3 \\ x^2y + xy^2 + \frac{1}{3}y^3 + \frac{1}{3}x^3 \end{pmatrix}, \qquad \boldsymbol{n}(x,y) = C_{Bnd}^{\frac{1}{2}}\begin{pmatrix} x \\ x+y \end{pmatrix}.$$

There are also non rigid body motions in the kernel of the $\boldsymbol{u} \mapsto \underline{\underline{\sigma}}(\boldsymbol{u})$. Thus the **strong** imposition of symmetry becomes important.

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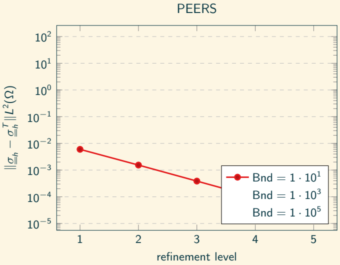

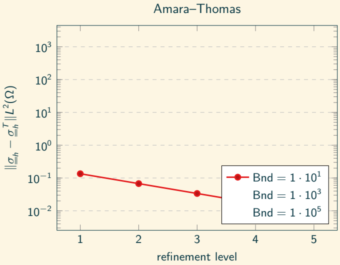

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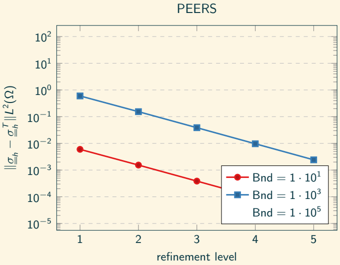

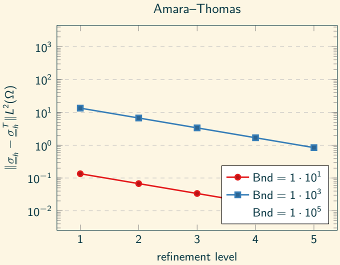

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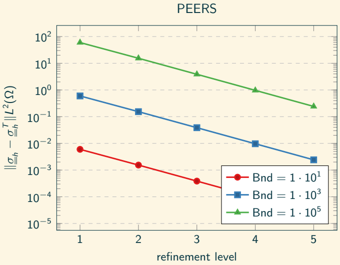

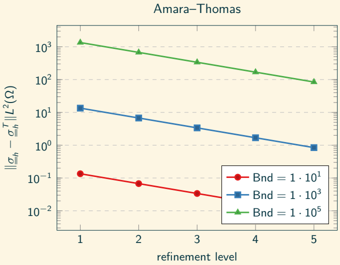

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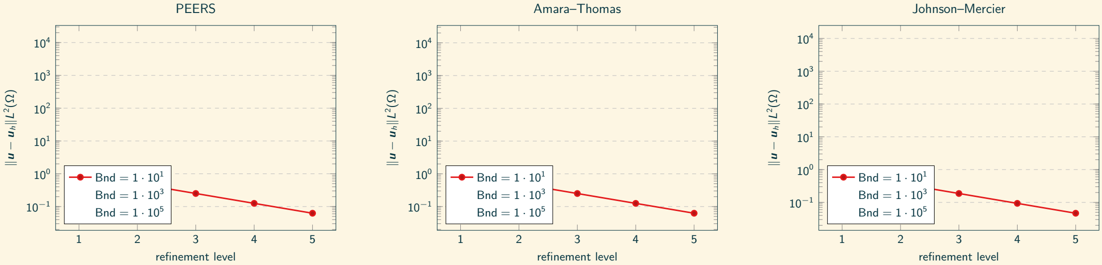

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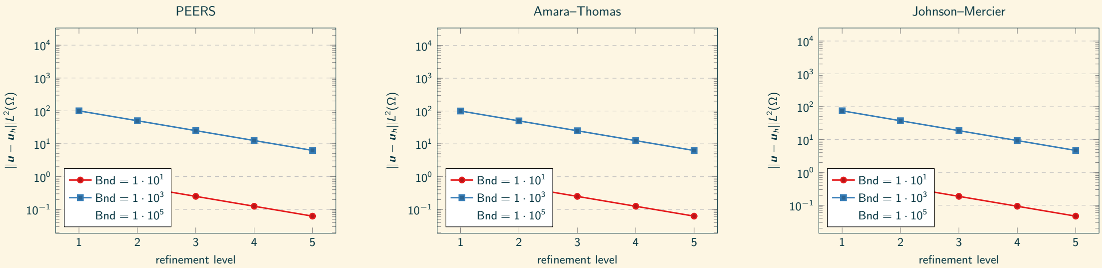

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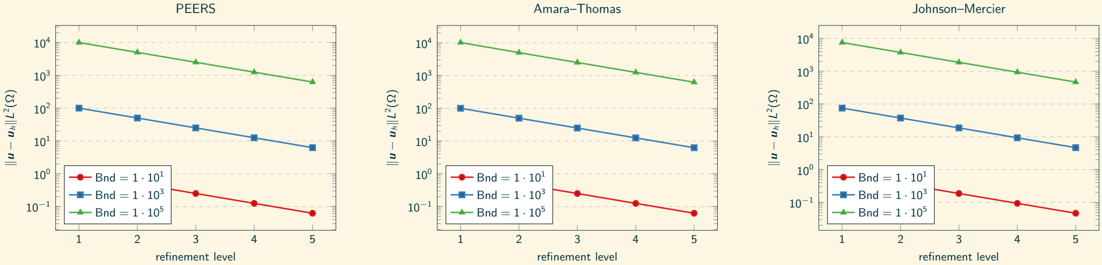

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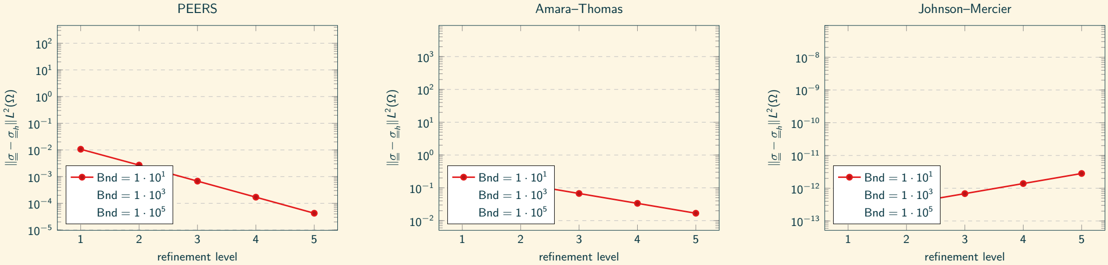

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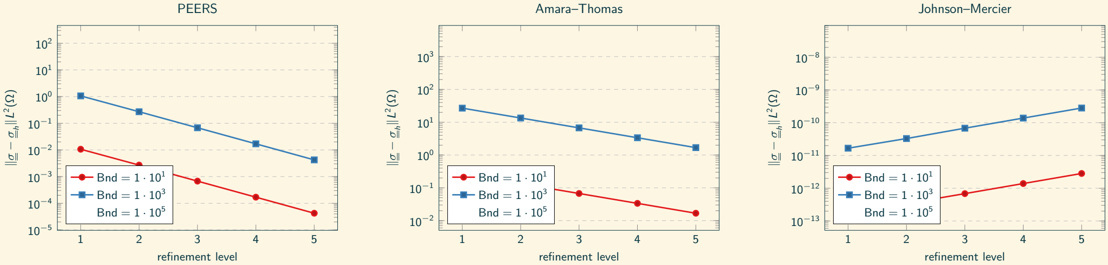

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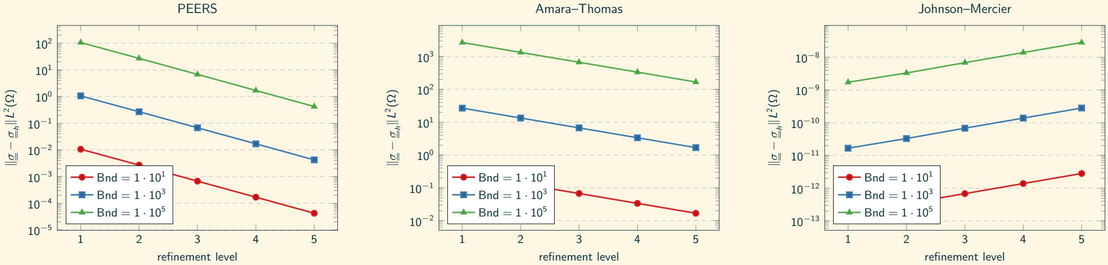

ARNOLD–FALK–WINTHER ELEMENTS – TRANSVERSE ANISOTROPY

D. Boffi, F. Brezzi and M. Fortin, **Mixed Finite Element Methods and Applications**, 2013.

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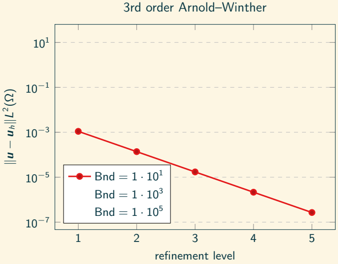

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$$\mathbb{S}_h = \mathcal{BDM}_k(\mathcal{T}_h)^{3r}, \qquad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega), \qquad \mathbb{W}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega).$$

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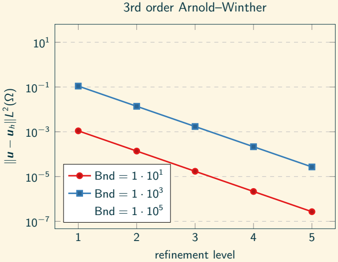

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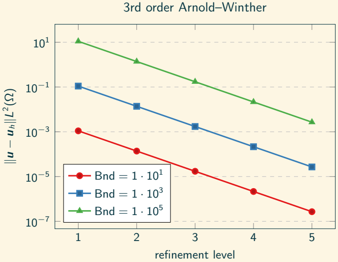

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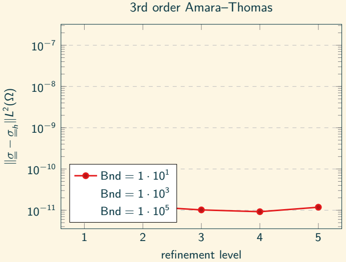

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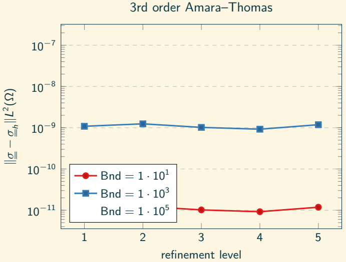

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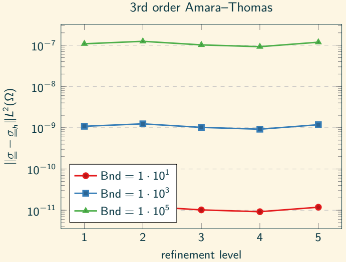

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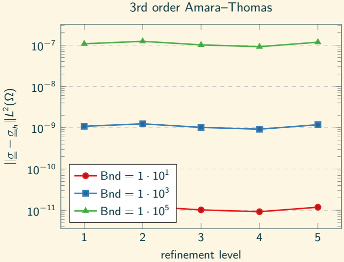

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SAINT–VENANT COMPATIBILITY CONDITION

The Saint–Venant compatibility condition is a necessary condition for $\underline{\underline{\epsilon}}(\boldsymbol{u})$ to be compatible with a displacement field $\boldsymbol{u}$, i.e.

$$\nabla \times \nabla \times \underline{\underline{\epsilon}} = 0.$$

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Saint–Venant compatibility condition: rank 1 tensors

Imposing the Saint–Venant compatibility condition when $\varepsilon(\boldsymbol{u}) = \boldsymbol{n} \otimes \boldsymbol{n}$, imposes the constraints that $\boldsymbol{n}$ is an **affine mapping**.

LIQUID CRYSTALS – ERICKSEN STRESS TENSOR

J. L. Ericksen, Conservation laws for liquid crystals. Transactions of the Society of Rheology, 1961.

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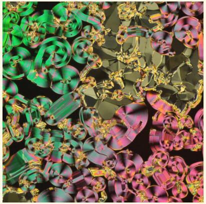

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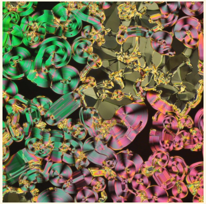

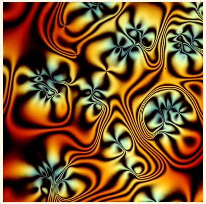

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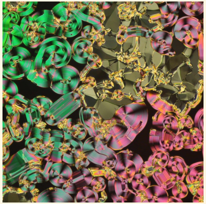

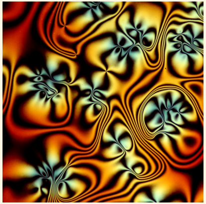

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Ericksen Stress Tensor

The Ericksen stress tensor is a symmetric rank 2 tensor, which is used to model the stress in liquid crystal materials, i.e.

$$\underline{\underline{\sigma}} = 2\nu \cdot \underline{\underline{\varepsilon}}(\boldsymbol{u}) + p\underline{\underline{I}} + K_F \cdot \nabla \boldsymbol{n}^T \nabla \boldsymbol{n}.$$

ERICKSEN FLUID – STRESS FORMULATION

We consider the following simplified Stokes problem with Ericksen stress tensor, i.e.

$$\begin{aligned} \frac{1}{\nu}\underline{\underline{\sigma}}^D - \nabla \boldsymbol{u} + \omega &= K_F \nabla \boldsymbol{n}^T \nabla \boldsymbol{n}, \\ \operatorname{div} \underline{\underline{\sigma}} - \nabla p &= -\boldsymbol{f}, \\ \underline{\underline{\sigma}} &= \underline{\underline{\sigma}}^T, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\boldsymbol{f}$ is once again the body force, ν is the fluid viscosity, and K_F is the Frank elastic constant.

ERICKSEN FLUID – WEAK FORMULATION

This problem can be written in weak form as follows:

$$\begin{aligned} a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) + b_2(\boldsymbol{u},\underline{\underline{\tau}}) &= \langle \underline{\underline{\tau}}\boldsymbol{n},\boldsymbol{g}\rangle_{\partial\Omega} && \forall \underline{\underline{\tau}} \in \mathbb{S}_h \\ b_2(\boldsymbol{v},\underline{\underline{\sigma}}) + b_1(\boldsymbol{v},p) &= -(\boldsymbol{f},\boldsymbol{v}), && \forall \boldsymbol{v} \in \mathbb{V}_h \\ b_1(\boldsymbol{u},q) &= \langle h, q\rangle_{\partial\Omega}, && \forall q \in \mathbb{Q}_h \end{aligned}$$

$$a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) := \frac{1}{2\mu}(\underline{\underline{\sigma}}^D,\underline{\underline{\tau}}^D)_{L^2(\Omega)}, \qquad b_1(\boldsymbol{u},q) := (\nabla\cdot\boldsymbol{u},p)_{L^2(\Omega)}, \qquad b_2(\boldsymbol{v},\underline{\underline{\sigma}}) := (\operatorname{div}\underline{\underline{\sigma}},\boldsymbol{v})_{L^2(\Omega)}$$

where $\mathbb{S}_h$, $\mathbb{V}_h$ and $\mathbb{Q}_h$ are appropriate finite element spaces for the stress, velocity and pressure, respectively.

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where $\mathbb{S}_h$, $\mathbb{V}_h$ and $\mathbb{Q}_h$ are appropriate finite element spaces for the stress, velocity and pressure, respectively.

To enforce the symmetry of the stress tensor, we can use introduce an additional Lagrange multiplier, i.e.

$$
c(\underline{\underline{\sigma}},\underline{\underline{\eta}}) := (\underline{\underline{\sigma}},\underline{\underline{\eta}})_{L^2(\Omega)} = 0 \qquad \forall \underline{\underline{\eta}} \in \mathbb{AS}_h,
$$

where $\mathbb{AS}_h$ is the space of antisymmetric tensors.

PATCH TEST – ERICKSEN FLUID

We here consider the following model problem, we pick

$$\boldsymbol{u} = C_u \begin{pmatrix} -cos(x)cosh(y) \\ sin(x)sinh(y) \end{pmatrix}, \qquad p = C_p sin(x)sinh(y),$$

$$\boldsymbol{n}(x,y) = C_n \begin{pmatrix} x \\ y \end{pmatrix}, \qquad K_F = sin(x)sinh(y).$$

We pick $C_n \gg 1$ and C_u, C_p such that $C_u + C_p + C_n = 0$, so that $\underline{\underline{\sigma}} \equiv 0$.

> There are also non polynomial in the kernel of the $\boldsymbol{u} \mapsto \underline{\underline{\sigma}}(\boldsymbol{u})$. Thus the **strong** imposition of symmetry becomes important.

THANK YOU!

On the symmetry constraint and angular momentum conservation in mixed stress formulation

Pablo Brubeck*, Charles Parker, II*, Umberto Zerbinati*