On the symmetry constraint and angular momentum conservation in mixed stress formulation

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CONTINUUM MECHANICS – BALANCE LAWS

The governing equations of continuum mechanics are the conservation are the conservation of mass, linear momentum, angular momentum.

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

$$\rho\Big(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot\underline{\underline{\sigma}} = \rho\boldsymbol{f},$$

$$\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta}\Big) - \nabla\cdot\underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho\tau},$$

where ρ is the density, $\boldsymbol{u}$ is the either the linear momentum, $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, $\boldsymbol{\eta}$ is the intrinsic angular momentum, $\boldsymbol{\xi}$ is the antisymmetric part of the Cauchy stress tensor, $\underline{\underline{\mu}}$ is the couple stress tensor, $\boldsymbol{f}$ is the body force, and $\boldsymbol{\tau}$ is the body torque.

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\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u}\cdot\nabla \boldsymbol{\eta}\Big) - \nabla\cdot \underline{\underline{\mu}} - \boldsymbol{\xi} &= \boldsymbol{\rho\tau},
\end{aligned}$$

where ρ is the density, $\boldsymbol{u}$ is the either the linear momentum, $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, $\boldsymbol{\eta}$ is the intrinsic angular momentum, $\boldsymbol{\xi}$ is the antisymmetric part of the Cauchy stress tensor, $\underline{\underline{\mu}}$ is the couple stress tensor, $\boldsymbol{f}$ is the body force, and $\boldsymbol{\tau}$ is the body torque.

The continuum mechanics governing equations need to be completed by **constitutive relations**.

Stokes Flow

A typical constitutive equation for the incompressible flow is the **Stokes flow**, which is given by

$$\underline{\underline{\sigma}} = 2\nu\underline{\underline{\varepsilon}}(\boldsymbol{u}) - p\underline{\underline{I}},$$

where ν is the kinematic viscosity, $\underline{\underline{\varepsilon}}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ is the strain rate tensor, and p is the Lagrange multiplier enforcing the incompressibility condition $\operatorname{div}\boldsymbol{u} = 0$.

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The Stokes flow is a linear problem, and it can be written in weak form as follows:

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \qquad b(\boldsymbol{u}, q) = 0,$$

where $a(\boldsymbol{u}, \boldsymbol{v}) = 2\nu(\underline{\underline{\varepsilon}}(\boldsymbol{u}), \underline{\underline{\varepsilon}}(\boldsymbol{v}))_{L^2(\Omega)}$ is the bilinear form associated with the viscous term, while $b(\boldsymbol{v}, p) = (\nabla \cdot \boldsymbol{v}, p)_{L^2(\Omega)}$ is the bilinear form for the incompressibility condition, and $(\boldsymbol{f}, \boldsymbol{v})$ is the linear form for the body force.

PRESSURE ROBUSTNESS – NO FLOW PROBLEM

J. V. Linke *et al.*, On the divergence constraint in mixed finite element methods for incompressible flows, **SIREV**, 2017.

A typical example used to demonstrate the pressure robustness exhibited by the divergence-free discretisations is the **no flow problem**, i.e.

$$\boldsymbol{f} = \begin{pmatrix} 0 \\ Ra(1-y+3y^2) \end{pmatrix}, \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad p = Ra\left(y^3 - \frac{1}{2}y^2 + y - \frac{7}{12}\right).$$

We expect the velocity to be independent of the pressure in the context of a divergence-free discretisation, contrary to the case of a non-divergence-free discretisation, i.e.

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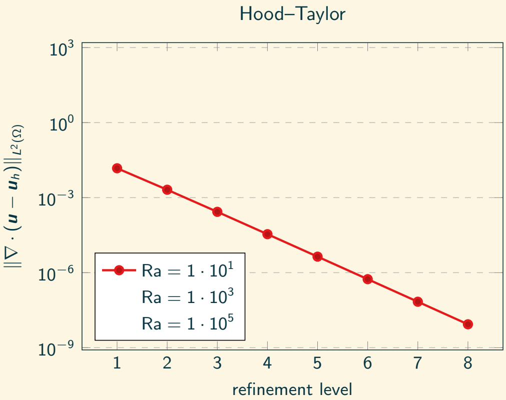

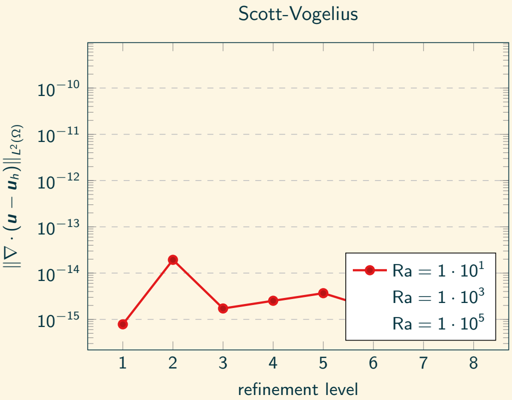

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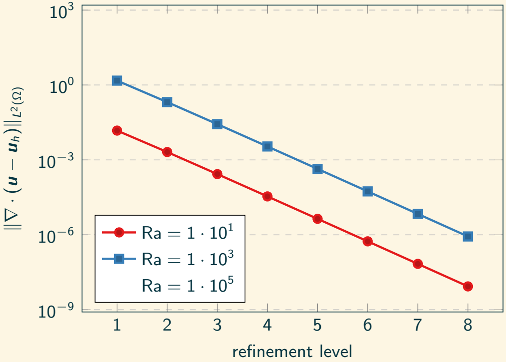

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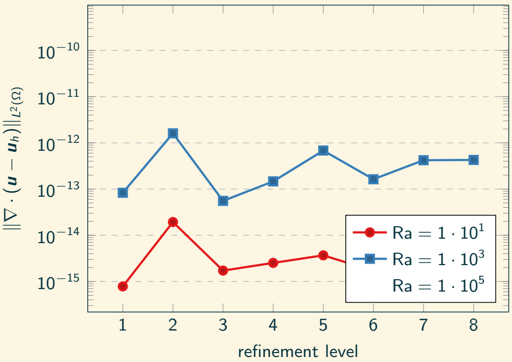

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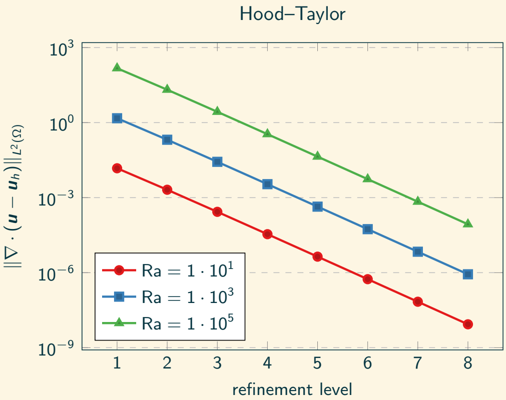

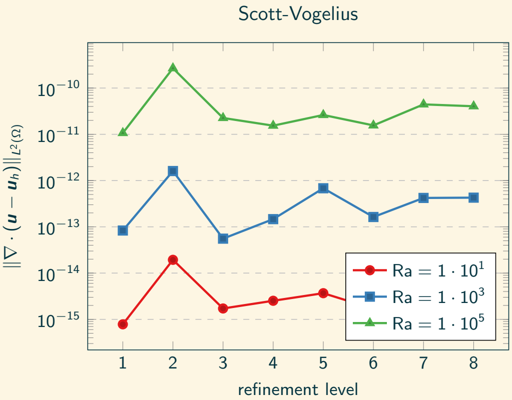

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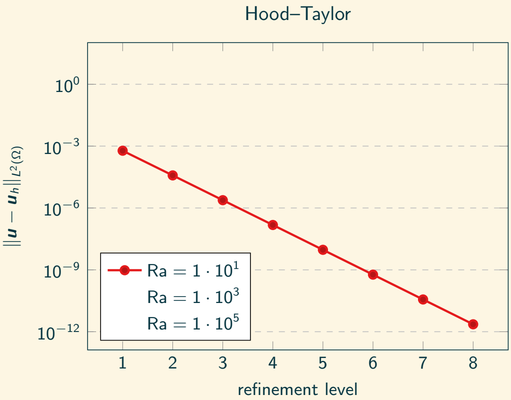

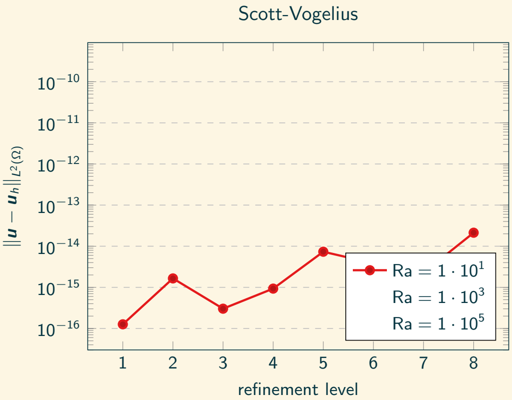

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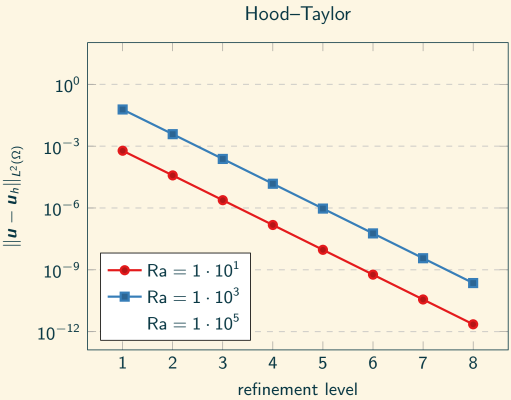

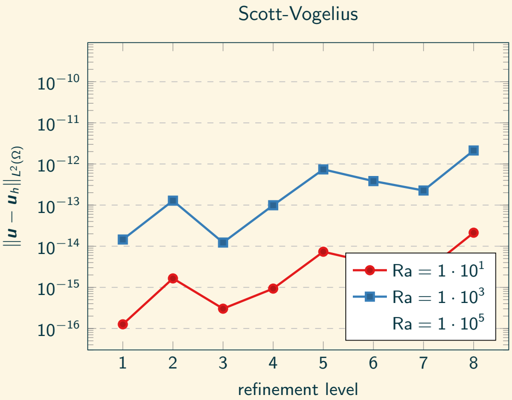

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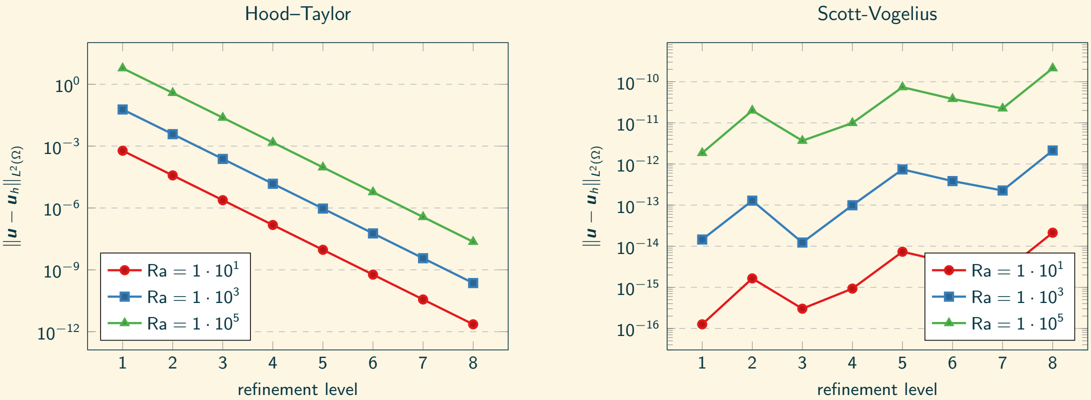

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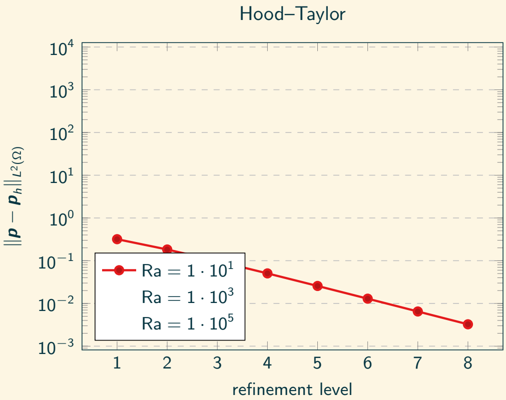

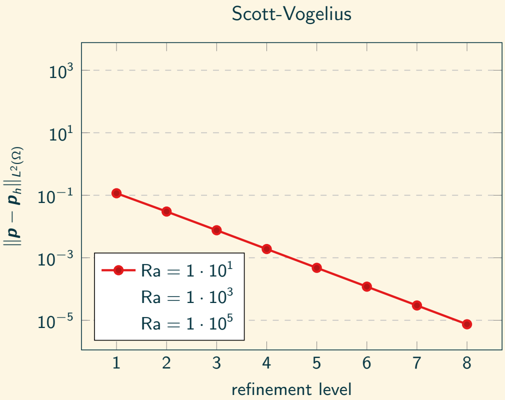

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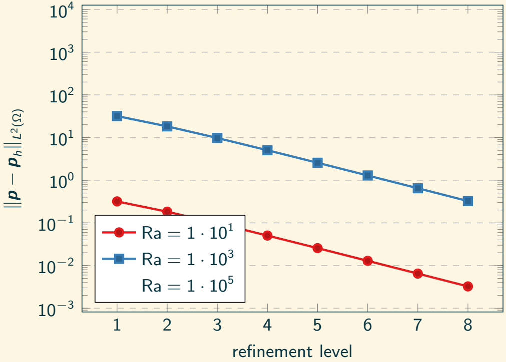

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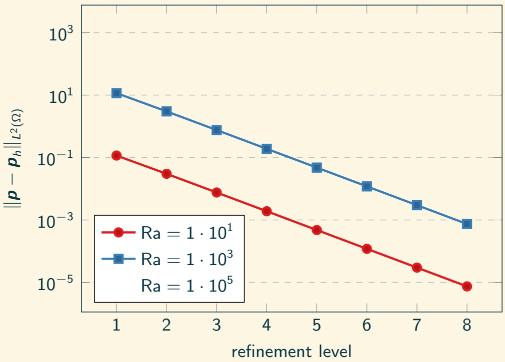

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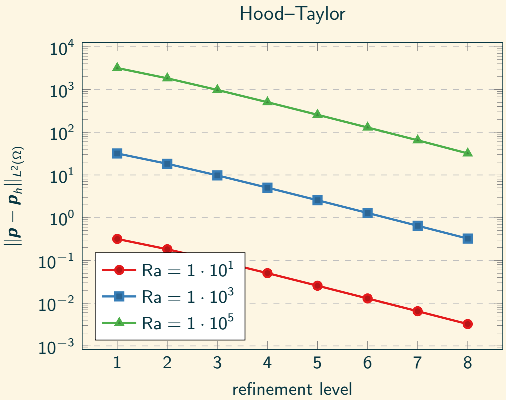

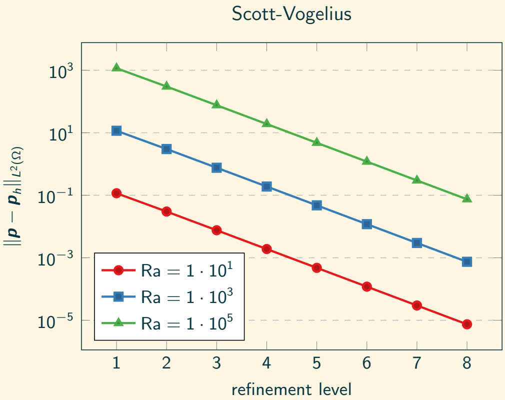

Let us begin considering a simpler yet related problem, namely the linear elasticity problem in stress formulation, i.e.

$$\operatorname{div} \underline{\underline{\sigma}} = \boldsymbol{f},$$
$$\underline{\underline{\sigma}} = 2\mu \underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda \operatorname{tr}(\underline{\underline{\varepsilon}}(\boldsymbol{u}))\underline{\underline{I}},$$

where $\boldsymbol{f}$ is once again the body force, μ is the shear modulus, λ is the first Lamé parameter.

This problem can be written in weak form as follows:

$$
\begin{aligned}
a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) + c(\underline{\underline{\tau}},\underline{\underline{\omega}}) + b(\boldsymbol{u},\underline{\underline{\tau}}) &= \langle \underline{\underline{\tau}}\boldsymbol{n},\boldsymbol{g}\rangle_{\partial\Omega} && \forall \underline{\underline{\tau}} \in \mathbb{S}_h \\
b(\boldsymbol{v},\underline{\underline{\sigma}}) &= (\boldsymbol{f},\boldsymbol{v}), && \forall \boldsymbol{v} \in \mathbb{V}_h \\
c(\underline{\underline{\sigma}},\underline{\underline{\eta}}) &= 0 && \forall \underline{\underline{\eta}} \in \mathbb{AS}_h,
\end{aligned}
$$

$$
a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) := \frac{1}{2\mu}(\underline{\underline{\sigma}}^D,\underline{\underline{\tau}}^D)_{L^2(\Omega)} + \frac{1}{d(d\lambda+2\mu)}(\mathrm{tr}(\underline{\underline{\sigma}}),\mathrm{tr}(\underline{\underline{\tau}}))_{L^2(\Omega)},
$$

$$
b(\boldsymbol{v},\underline{\underline{\sigma}}) := (\mathrm{div}\,\underline{\underline{\sigma}},\boldsymbol{v})_{L^2(\Omega)}, \qquad c(\underline{\underline{\sigma}},\underline{\underline{\eta}}) := (\underline{\underline{\sigma}},\underline{\underline{\eta}})_{L^2(\Omega)}
$$

where the superscript D denotes the deviatoric part of a tensor, i.e. $\underline{\underline{\sigma}}^D = \underline{\underline{\sigma}} - \frac{1}{d}\,\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}$ and $\mathbb{AS}_h$ is the space of antisymmetric tensors.

PATCH TEST – RIGID BODY MOTION

We begin from the most simple scenario, i.e. we try to induce a large component in the antisymmetric part of the stress tensor, via rigid body motion.

$$\boldsymbol{u} = C_{Bnd}\begin{pmatrix} -y \\ x \end{pmatrix}, \qquad \underline{\underline{\sigma}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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The exact solution are in the discrete spaces $[\mathbb{P}_1(\mathcal{T}_h)]^2$ and $[\mathbb{P}_0(\mathcal{T}_h)]^{2\times 2}$, hence $\underline{\underline{\eta}}$ can be approximated exactly by a **“low-order”** finite element approximation.

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The only elements in the kernel of the symmetric part of the gradient are the rigid body motions.

SYMMETRY CONSTRAINT – RIGID BODY MOTION

D. N. Arnold, *et al.*, PEERS: a new mixed finite element for plane elasticity, **JJIAM**, 1984,

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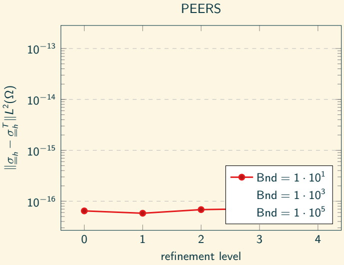

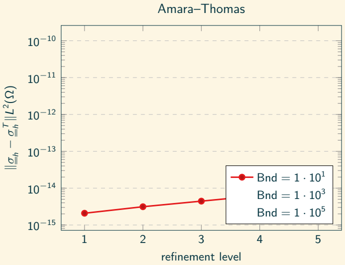

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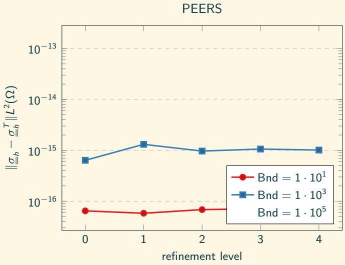

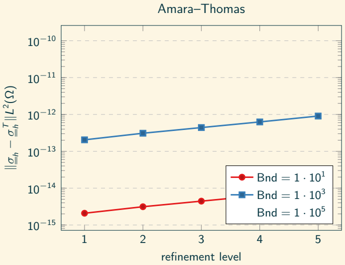

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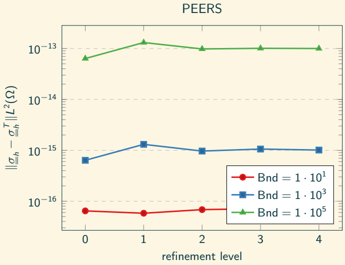

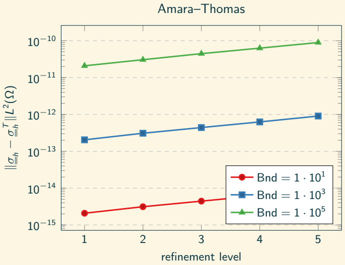

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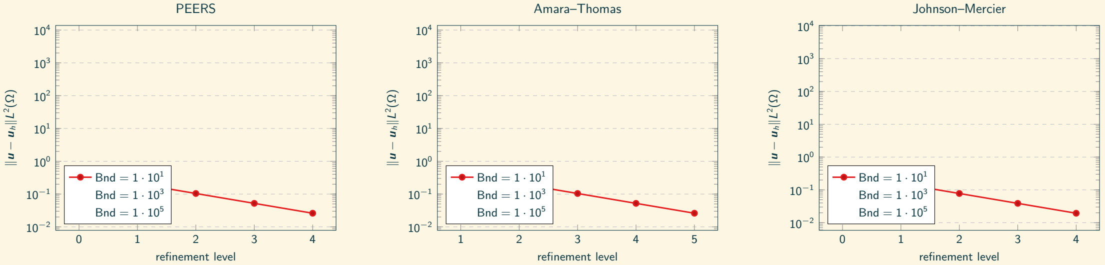

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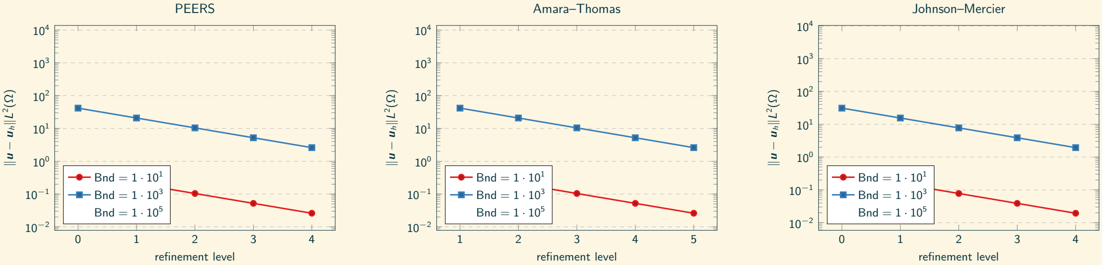

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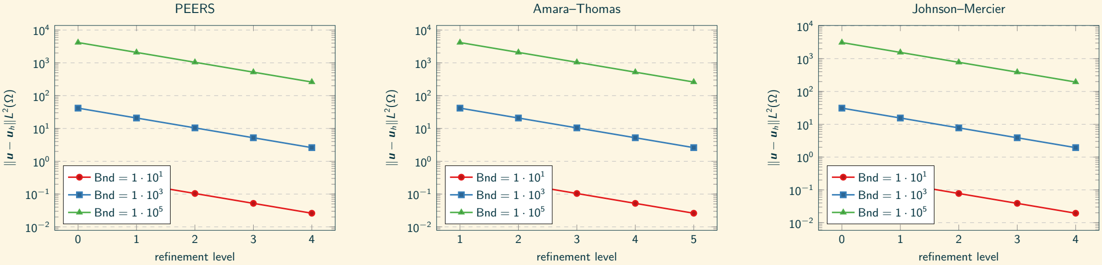

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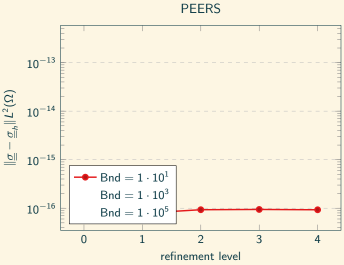

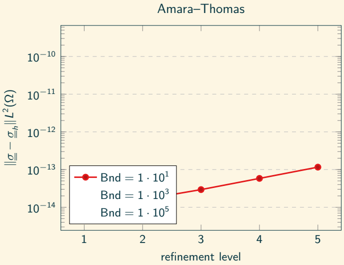

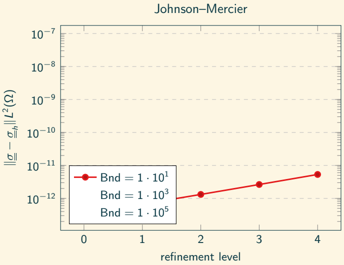

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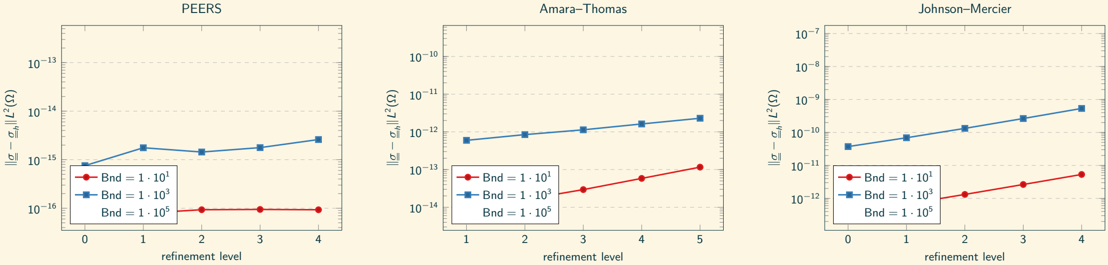

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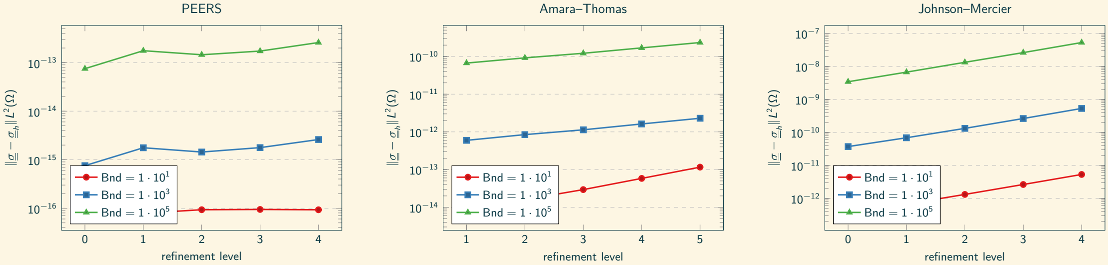

LIQUID CRYSTAL POLYMER NETWORKS – TRANSVERSE ANISOTROPY

T. .J. White, Photomechanical Effects in Liquid–Crystalline Polymer Networks and Elastomers, **J. Polymer Science**, 2017

R. H. Nochetto *et al.*, Convergent FEM for a Membrane Model of Liquid Crystal Polymer Networks, **SINUM**, 2023.

A liquid crystal polymer network (LCNs) is a material are **polymers** that exhibit a liquid crystalline phase, and are **crosslinked** to form a network structure, to obtain a material with unique mechanical properties. The most prominent example is **kevlar**.

Transversly Isotropic Material

LCNs exhibit a **transverse isotropy** in their mechanical properties, i.e. we can express the stress tensor as

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda(\nabla\cdot\boldsymbol{u})\underline{\underline{I}} + \boldsymbol{n}\otimes\boldsymbol{n}.$$

PATCH TEST – TRANSVERSE ANISOTROPY

We here consider the following model problem, we pick

$$\boldsymbol{u} = -\frac{C_{Bnd}}{2\mu}\begin{pmatrix} \frac{1}{3}x^3 - \frac{2}{3}y^3 \\ x^2y + xy^2 + \frac{1}{3}y^3 + \frac{1}{3}x^3 \end{pmatrix}, \qquad \boldsymbol{n}(x,y) = C_{Bnd}^{\frac{1}{2}}\begin{pmatrix} x \\ x+y \end{pmatrix}.$$

There are also non rigid body motions in the kernel of the $\boldsymbol{u} \mapsto \underline{\underline{\sigma}}(\boldsymbol{u})$. Thus the **strong** imposition of symmetry becomes important.

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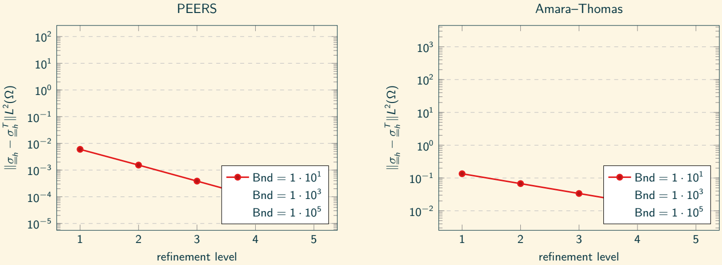

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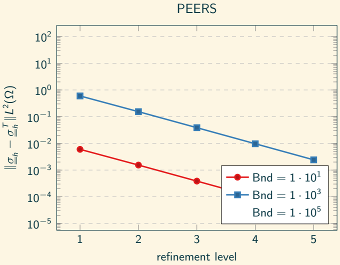

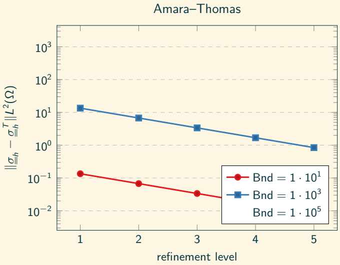

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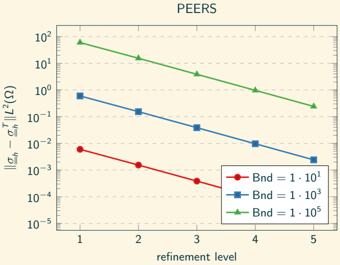

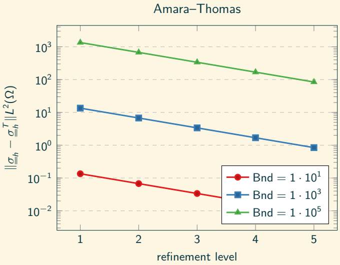

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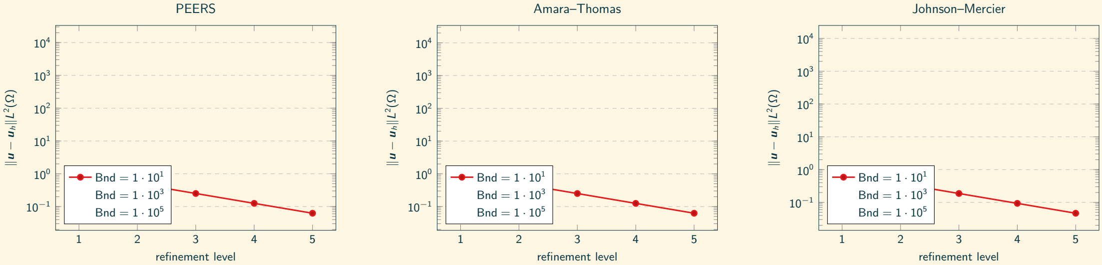

SYMMETRY CONSTRAINT – TRANSVERSE ANISOTROPY

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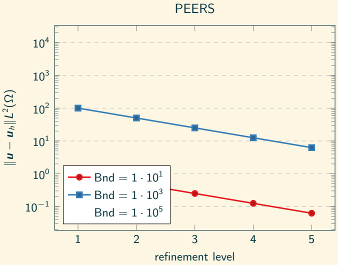

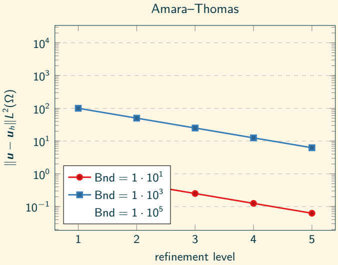

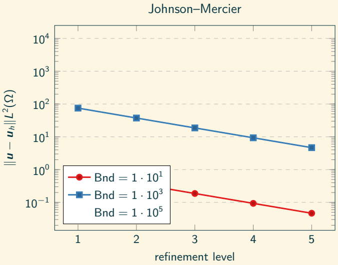

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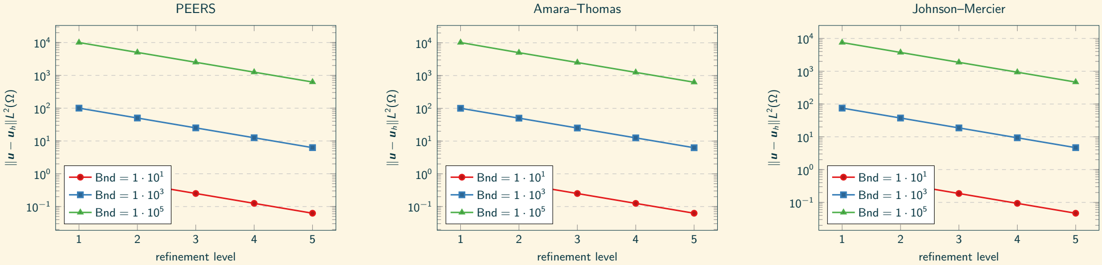

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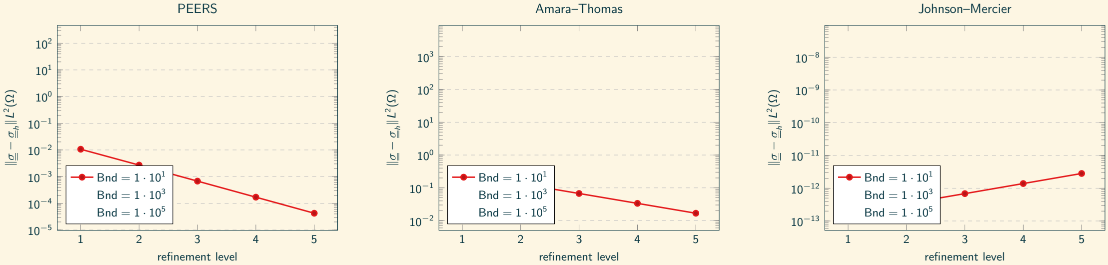

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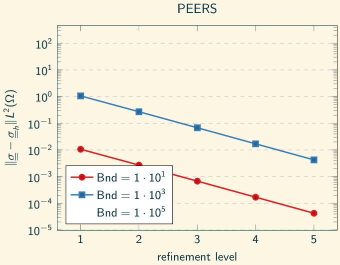

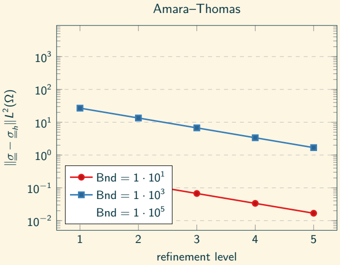

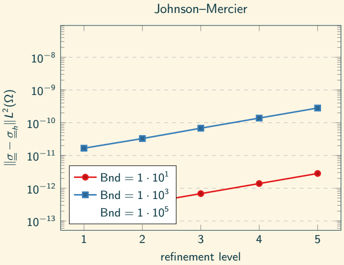

SYMMETRY CONSTRAINT – TRANSVERSE ANISOTROPY

Mathematical Institute

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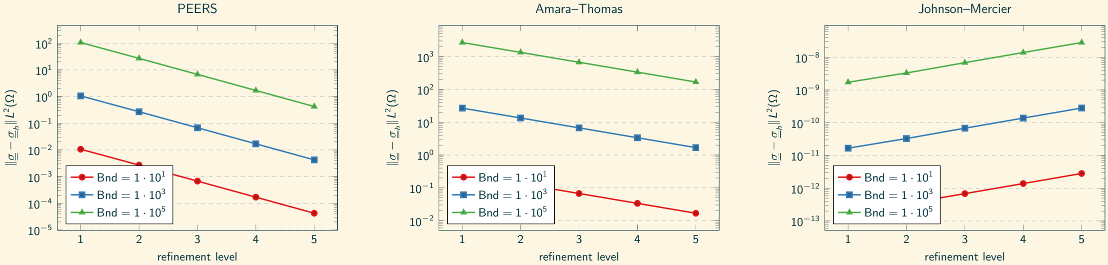

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We can compute the exact off diagonal entries of $\underline{\underline{\eta}}$, i.e.

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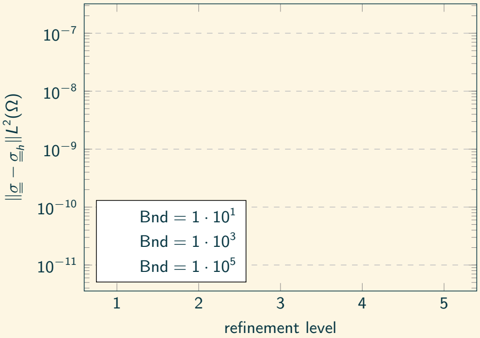
3rd order Amara–Thomas



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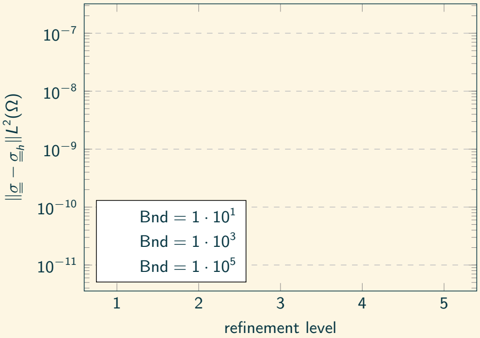

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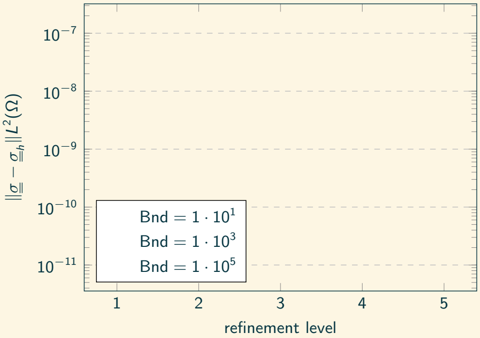

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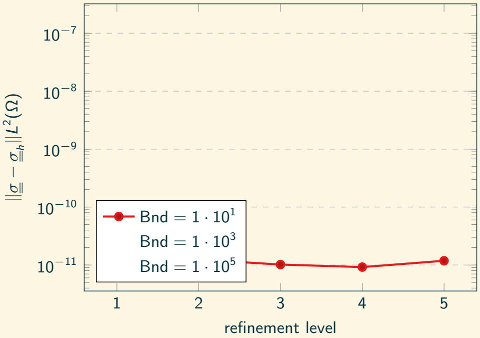

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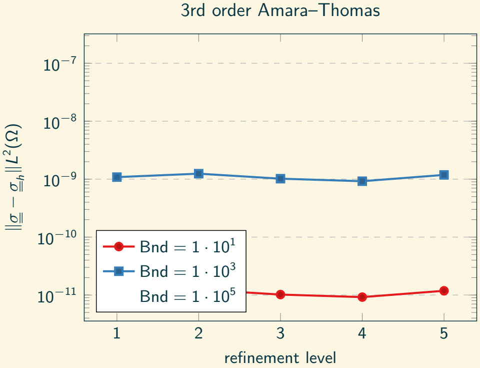

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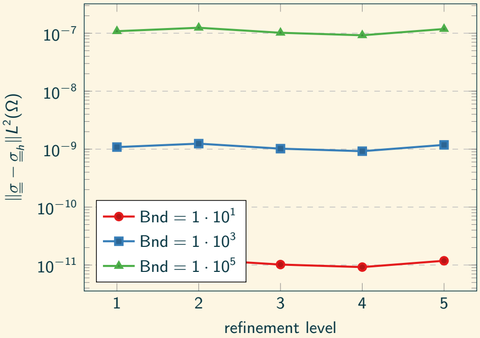

LIQUID CRYSTALS – ERICKSEN STRESS TENSOR

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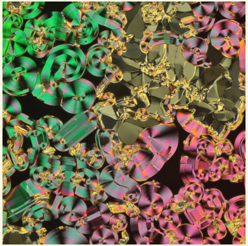

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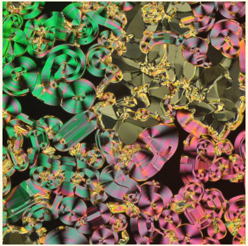

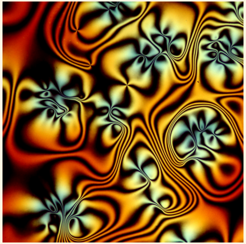

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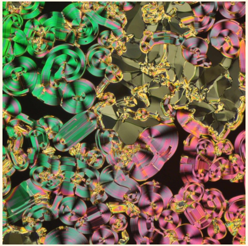

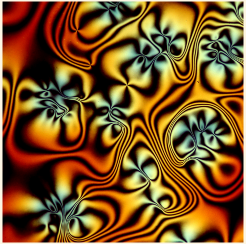

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Ericksen Stress Tensor

The Ericksen stress tensor is a symmetric rank 2 tensor, which is used to model the stress in liquid crystal materials, i.e.

$$\underline{\underline{\sigma}} = 2\nu \cdot \underline{\underline{\varepsilon}}(\boldsymbol{u}) + p\underline{\underline{I}} + K_F \cdot \nabla \boldsymbol{n}^T \nabla \boldsymbol{n}.$$

ERICKSEN FLUID – STRESS FORMULATION

UNIVERSITY OF OXFORD
Mathematical Institute

We consider the following simplified Stokes problem with Ericksen stress tensor, i.e.

$$
\begin{aligned}
\frac{1}{\nu}\underline{\underline{\sigma}}^D - \nabla \boldsymbol{u} + \omega &= K_F \nabla \boldsymbol{n}^T \nabla \boldsymbol{n},\\
\operatorname{div} \underline{\underline{\sigma}} - \nabla p &= -\boldsymbol{f},\\
\underline{\underline{\sigma}} &= \underline{\underline{\sigma}}^T,\\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned}
$$

where $\boldsymbol{f}$ is once again the body force, ν is the fluid viscosity, and K_F is the Frank elastic constant.

ERICKSEN FLUID – WEAK FORMULATION

This problem can be written in weak form as follows:

$$
\begin{aligned}
a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) + b_2(\boldsymbol{u}, \underline{\underline{\tau}}) &= \langle \underline{\underline{\tau}}\boldsymbol{n}, \boldsymbol{g}\rangle_{\partial\Omega} && \forall \underline{\underline{\tau}} \in \mathbb{S}_h\\
b_2(\boldsymbol{v}, \underline{\underline{\sigma}}) + b_1(\boldsymbol{v}, p) &= -(\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in \mathbb{V}_h\\
b_1(\boldsymbol{u}, q) &= 0, && \forall q \in \mathbb{Q}_h
\end{aligned}
$$

$$
a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) := \frac{1}{2\mu}(\underline{\underline{\sigma}}^D, \underline{\underline{\tau}}^D)_{L^2(\Omega)}, \qquad b_1(\boldsymbol{u}, q) := (\nabla \cdot \boldsymbol{u}, p)_{L^2(\Omega)}, \qquad b_2(\boldsymbol{v}, \underline{\underline{\sigma}}) := (\operatorname{div} \underline{\underline{\sigma}}, \boldsymbol{v})_{L^2(\Omega)}
$$

where $\mathbb{S}_h$, $\mathbb{V}_h$ and $\mathbb{Q}_h$ are appropriate finite element spaces for the stress, velocity and pressure, respectively.

ERICKSEN FLUID – WEAK FORMULATION

This problem can be written in weak form as follows:

$$\begin{aligned} a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) + b_2(\boldsymbol{u}, \underline{\underline{\tau}}) &= \langle \underline{\underline{\tau}}\boldsymbol{n}, \boldsymbol{g}\rangle_{\partial\Omega} && \forall \underline{\underline{\tau}} \in \mathbb{S}_h \\ b_2(\boldsymbol{v}, \underline{\underline{\sigma}}) + b_1(\boldsymbol{v}, p) &= -(\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in \mathbb{V}_h \\ b_1(\boldsymbol{u}, q) &= 0, && \forall q \in \mathbb{Q}_h \end{aligned}$$

$$a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) := \frac{1}{2\mu}(\underline{\underline{\sigma}}^D, \underline{\underline{\tau}}^D)_{L^2(\Omega)}, \qquad b_1(\boldsymbol{u}, q) := (\nabla \cdot \boldsymbol{u}, p)_{L^2(\Omega)}, \qquad b_2(\boldsymbol{v}, \underline{\underline{\sigma}}) := (\operatorname{div} \underline{\underline{\sigma}}, \boldsymbol{v})_{L^2(\Omega)}$$

where $\mathbb{S}_h$, $\mathbb{V}_h$ and $\mathbb{Q}_h$ are appropriate finite element spaces for the stress, velocity and pressure, respectively.

To enforce the symmetry of the stress tensor, we can use introduce an additional Lagrange multiplier, i.e.

$$c(\underline{\underline{\sigma}}, \underline{\underline{\eta}}) := (\underline{\underline{\sigma}}, \underline{\underline{\eta}})_{L^2(\Omega)} = 0 \qquad \forall \underline{\underline{\eta}} \in \mathbb{AS}_h,$$

where $\mathbb{AS}_h$ is the space of antisymmetric tensors.

PATCH TEST – ERICKSEN FLUID

We here consider the following model problem, we pick

$$\boldsymbol{u} = C_u \begin{pmatrix} -cos(x)cosh(y) \\ sin(x)sinh(y) \end{pmatrix}, \qquad p = C_p sin(x)cosh(y),$$

$$\boldsymbol{n}(x,y) = C_n \begin{pmatrix} x \\ y \end{pmatrix}, K_F = sin(x)sinh(y).$$

We pick $C_n >> 1$ and C_u, C_p such that $C_u + C_p + C_K = 0$, so that $\underline{\underline{\sigma}} \equiv 0$.

There are also non polynomial in the kernel of the $\boldsymbol{u} \mapsto \underline{\underline{\sigma}}(\boldsymbol{u})$. Thus the **strong** imposition of symmetry becomes important.

ERICKSEN TENSOR - PATCH TEST

We now design a patch test, for the **intrinsic** angular momentum, i.e.

$$\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta}\Big) - \nabla\cdot\underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho\tau},$$

We now design a patch test, for the **intrinsic** angular momentum, i.e.

$$\rho\Big(\partial_t \boldsymbol{\eta} + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta}\Big) - \nabla \cdot \underline{\underline{\mu}} - \boldsymbol{\xi} = \boldsymbol{\rho\tau},$$

We pick a very silly couple stress tensor, i.e. $\underline{\underline{\mu}} = \nabla \boldsymbol{\eta}$, assume that η vanish at the boundary and have zero torque, i.e. $\boldsymbol{\tau} \equiv 0$.

CONSERVATION OF ANGULAR MOMENTUM – ERICKSEN TENSOR

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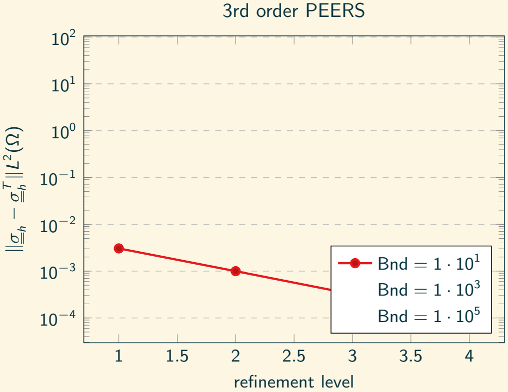

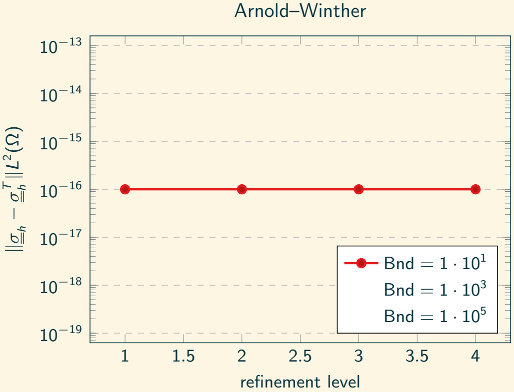

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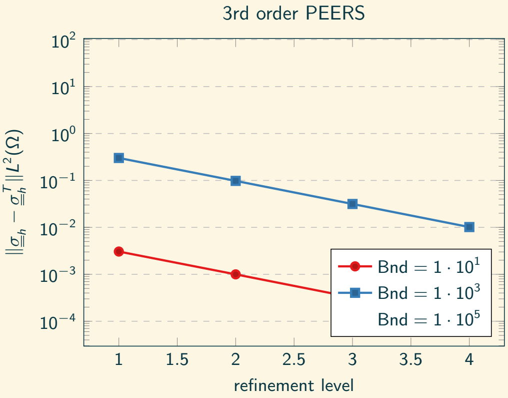

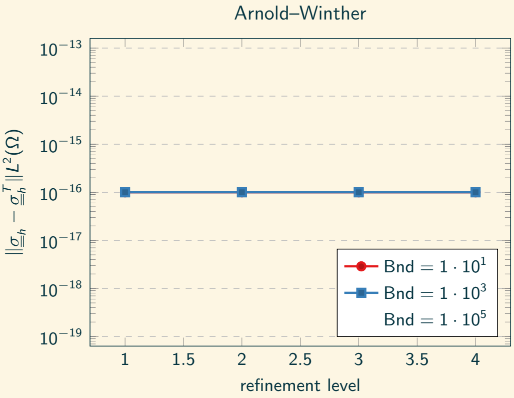

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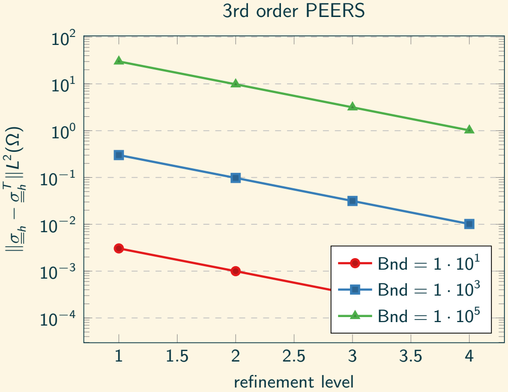

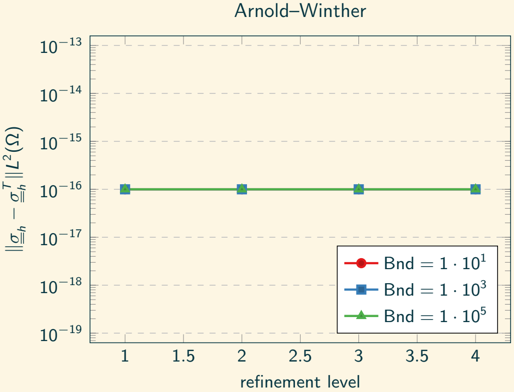

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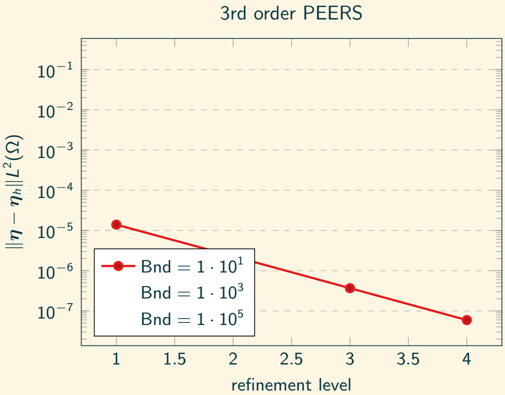

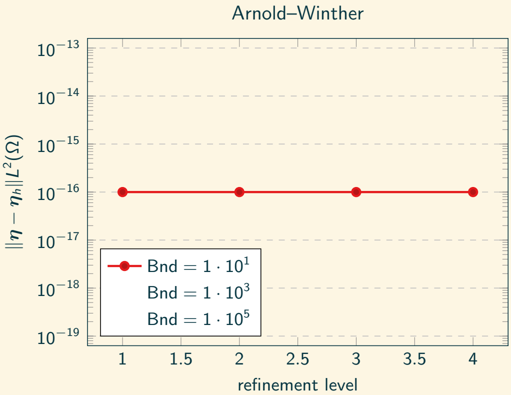

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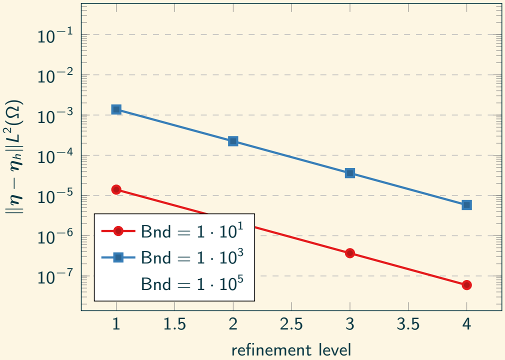
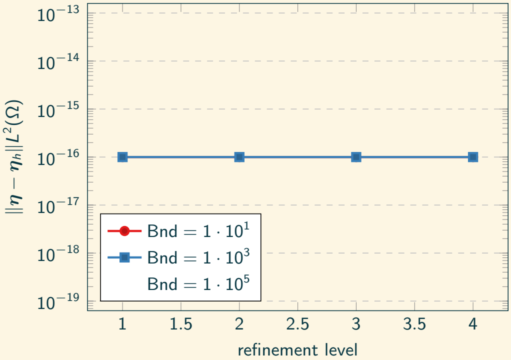
3rd order PEERS



Arnold–Winther



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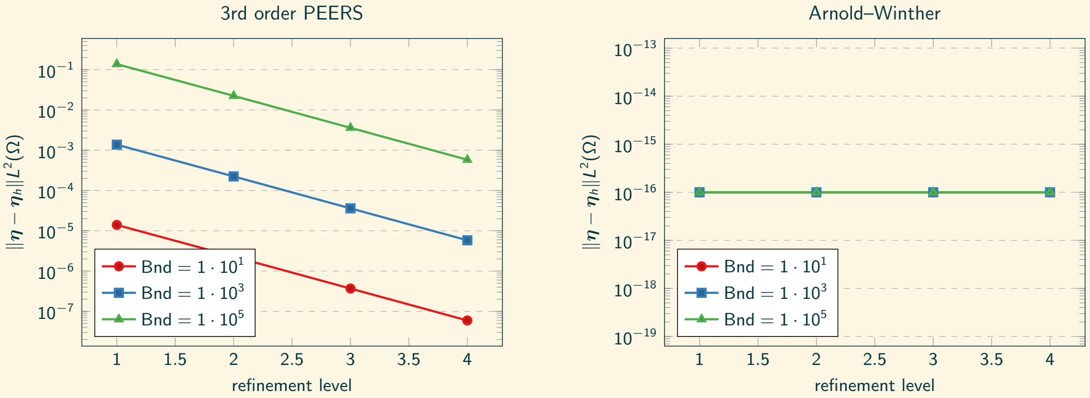

THANK YOU!

On the symmetry constraint and angular momentum conservation in mixed stress formulation

Pablo Brubeck*, Charles Parker, II*, Umberto Zerbinati*

SOME WEAKLY SYMMETRIC MIXED FINITE ELEMENTS

PEERS

$$\mathbb{S}_h = \mathcal{RT}_k(\mathcal{T}_h)^{3r}, \qquad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega), \qquad \mathbb{W}_h = \mathcal{P}_k(\mathcal{T}_h) \cap H^1(\Omega).$$

Arnold–Falk–Winther

$$\mathbb{S}_h = \mathcal{BDM}_k(\mathcal{T}_h)^{3r}, \qquad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega), \qquad \mathbb{W}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega).$$

Amara–Thomas

$$\mathbb{S}_h = \mathcal{BDFM}_k(\mathcal{T}_h)^{3r}, \qquad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega), \qquad \mathbb{W}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega).$$

When $k = 1$, notice that $\mathcal{BDFM}_1(\mathcal{T}_h)^{3r} = \mathcal{BDM}_1(\mathcal{T}_h)^{3r}$, thus this element is equivalent to the Arnold–Falk–Winther element of order 1.

SOME STRONGLY SYMMETRIC MIXED FINITE ELEMENTS

Arnold–Winther

$$\mathbb{S}_h = \mathcal{AW}_k(\mathcal{T}_h), \quad \mathbb{V}_h = \mathcal{P}_{k-2}(\mathcal{T}_h) \cap L^2(\Omega).$$

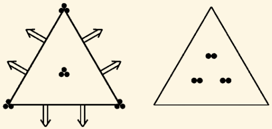

Figure: Arnold–Winther element of order $k = 3$ on a triangular mesh.

Johnson–Mercier

$$\mathbb{S}_h = \mathcal{JM}_k(\mathcal{T}_h), \quad \mathbb{V}_h = \mathcal{P}_{k-1}(\mathcal{T}_h) \cap L^2(\Omega).$$

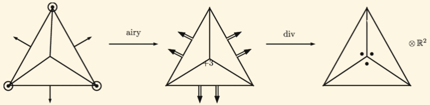

Figure: The complex leading to the Johnson–Mercier element of order $k = 1$ on a Alfeld mesh.

SYMMETRY CONSTRAINT – A PRIORI ERROR ESTIMATE

When reduced symmetry is imposed, the error estimate for the discrete scheme is fully coupled and take the form

$$\begin{aligned}\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} + \mu\beta_h\Big[\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\underline{\underline{\eta}} - \underline{\underline{\eta}}_h\|_{L^2(\Omega)}\Big] &\leq C\beta_h^{-1}\inf_{\tau_h\in\mathbb{S}_h}\|\underline{\underline{\sigma}} - \tau_h\|_{L^2(\Omega)} \\ &+ C\mu\inf_{v_h\in\mathbb{V}_h}\|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Omega)} \\ &+ C\mu\inf_{\eta_h\in\mathbb{AS}_h}\|\underline{\underline{\eta}} - \eta_h\|_{L^2(\Omega)}.\end{aligned}$$

SYMMETRY CONSTRAINT – A PRIORI ERROR ESTIMATE

When reduced symmetry is imposed, the error estimate for the discrete scheme is fully coupled and take the form

$$
\begin{aligned}
\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} + \mu\beta_h \Big[\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\underline{\underline{\eta}} - \underline{\underline{\eta}}_h\|_{L^2(\Omega)} \Big] \leq\ & C\beta_h^{-1} \inf_{\tau_h \in \mathbb{S}_h} \|\underline{\underline{\sigma}} - \tau_h\|_{L^2(\Omega)} \\
& + C\mu \inf_{v_h \in \mathbb{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Omega)} \\
& + C\mu \inf_{\eta_h \in \mathbb{AS}_h} \|\underline{\underline{\eta}} - \eta_h\|_{L^2(\Omega)}.
\end{aligned}
$$

Strong Symmetry

If we impose the symmetry constraint and $\nabla \cdot \mathbb{S}_h = \mathbb{V}_h$, we obtain a decoupled error estimate of the form

$$
\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h\|_{L^2(\Omega)} \leq C\beta_h^{-1} \inf_{\tau_h \in \mathbb{S}_h} \|\underline{\underline{\sigma}} - \underline{\underline{\tau}}_h\|_{L^2(\Omega)}.
$$