PART B: The Rayleigh quotient.

* The main and only difference with ERN/GUERMOND is that we consider a generic $m: X\times X \to \mathbb{R}$.

In this section we will work under the assumption that $a: X\times X \to \mathbb{R}$ and $m: X\times X \to \mathbb{R}$ are SYMMETRIC POSITIVE DEFINITE bilinear form, i.e. it $\exists \alpha>0$ and $\mu>0$ such that

$$\mu \le m(u,u) \quad \text{and} \quad \alpha \le a(u,u) \qquad \forall u \in X,$$

and $a: X\times X \to \mathbb{R}$ and $m: X\times X \to \mathbb{R}$ is bounded.

We consider the usual variational eigenvalue problem

$$\text{Find } (\lambda, u) \in \mathbb{C}\times X \text{ such that } a(u,v) = \lambda\, m(u,v) \quad \forall v \in X \qquad \text{(EQ 1)}$$

In the previous part we caracterised ω as the reciprocal eigenvalues of a compact self adjoint operator when $m: X\times X \to \mathbb{R}$ was the ℓ^2 scalar product. We notice that the same characterisation also hold for any $m: X\times X \to \ell^2$ that is symmetric, in fact the operator

$$T: X \longrightarrow X$$
$$f \mapsto Tf \text{ s.t. } a(Tf, v) = m(f,v) \qquad \forall v \in X,$$

by virtue of the LAX-MILGRAM lemma and the fact that both $m: X\times X \to \mathbb{R}$ and $a: X\times X \to \mathbb{R}$ induce a scalar product.

REMARK You should not be worried about the operator m, because it appears every time we actually assemble the load-vector for the FEM method and at discrete level it takes care of passing from nodal value to the load vector i.e. an element of the dual of X.

One theorem that we didn't mention before is the HILBERT BASIS theorem, i.e.

THEOREM Let $T: X \to X$ be a compact self-adjoint operator, then the eigenfunction of T, form a basis for the space X, provided that X is a separable Hilbert space. In particular such basis is orthogonal wrt $m: X\times X \to \mathbb{R}$.

DEFINITION The Rayleigh quotient of a function $v \in X$, relative to the bilinear forms $a: X\times X \to \mathbb{R}$ and $m: X\times X \to \mathbb{R}$ is defined as, $R(v) := \dfrac{a(v,v)}{m(v,v)}$, where m is a POSITIVE-DEFINITE bilinear form.

Armed with the notion of a Rayleigh quotient we can characterise the problem of solving the eigenvalue problem, (EQ 1) as a CALCULUS OF VARIATION PROBLEM.

PROPOSITION Let λ_1 be the smallest eigenvalue of (EQ 3) and u_1 be the corresponding eigen-vector, then:

$$\lambda_1 = R(u_1) = \min_{u\in X} R(u).$$

where

PROOF Notice that if u_1 is the first eigenfunction of (EQ 1) then

$$a(u_1,u_1) = \lambda\, m(u_1,u_1) \Rightarrow \lambda_1 = \frac{a(u_1,u_1)}{m(u_1,u_1)} = R(u_1).$$

Abviusly we have for free that $\lambda_1 \ge \inf_{u\in X} R(u)$ we need to prove the converse.

We begin expanding a vector $v \in X$ is the Hilbert basis generated by the eigenvalue

$\underline{v} = \sum_{n=1}^{\infty} v^{(n)} \underline{u}_n$, substituting this in the Rayleigh quotient we get that:

$$R(v) = \frac{a(v,v)}{m(v,v)} = \frac{\sum_{k=1}^{\infty} v^{(k)} a(u_k,u_k)}{\sum_{k=1}^{\infty} v^{(k)} m(u_k,u_k)} \geq \frac{v^{(1)} a(u_1,u_1)}{v^{(1)} m(u_1,u_1)} = \lambda_1 \quad \square$$

Notice that this characterisation doesn't only hold for the first eigenvalue but choosing appropriately the space over which we are minimizing we can characterize all eigenvalues as a min/max problem.

PROPOSITION For all $m \geq 1$ we have that and denoting (λ_i, u_i) the eigenvalue and eigen functions of (EQ 1) we have that,

$$\lambda_m = \max_{E_m \subseteq X} \min_{v \in E_m^{\perp}} R(v)$$, where E_m denotes the subspace of dimension m and $E_m^{\perp}$ its orthogonal wrt to $m: X \times X \to \mathbb{R}$. Clearly the eigen function $u_m \in \{u_1, \dots, u_m\}$ hence we have:

$$\min_{E_m} \max_{v \in E_m} R(v) \leq \max_{v \in \{u_1,\dots,u_m\}} R(v) = \lambda_m \text{ since } R(u_i) = \lambda_i .$$

Now let us pick any $E_m \subseteq X$ with $\dim(E_m) = m-1$, let us now consider the orthogonal projection of E_m onto $\{u_1, \dots, u_{m-1}\}$ by the RANK NULLITY THEOREM there is a non zero element in the kernel of the projection, we call it w and observe $w \in W_{m-1}^{\perp}$ hence we have $w = \sum_{n \geq m} w^{(n)} u_m$ it is simple to show that

$$R(w) = \frac{\sum_{n=1}^{\infty} u^{(n)} a(u_n,u_n)}{\sum_{n=1}^{\infty} u^{(n)} m(u_n,u_n)} \geq \lambda_m$$

$$\min_{E_m} \max_{v \in E_m} R(v) \geq \lambda_m$$ so we can conclude.

We now introduce the discrete projection operator as we have done for the continuous solution operator, i.e.

$$T_h : X \longrightarrow X_h \subseteq X$$
$$f \mapsto Tf \text{ s.t. } a(Tf, v_h) = a(f, v_h) \quad \forall v_h \in X_h$$

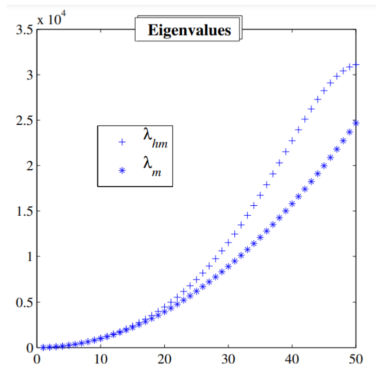
which once again we know its well posed by CONFORMITY, i.e. $X_h \subseteq X$ and LAX-MILGRAM LEMMA. We also define the quantity, $\sigma_h^{(m)} = \min_{\substack{v \in \{u_1,\dots,u_m\} \\ v \neq 0}} \frac{m(T_h v, T_h v)}{\sqrt{m(v,v)}}$

We now consider the discrete variational eigenvalue problem:

Find $u_h \in X_h$ such that $a(u_h, v_h) = \lambda^h m(u_h, v_h) \quad \forall v_h \in V_h$. (EQ 2)

Notice now that by the conformity of the approximation, i.e. $X_h \subseteq X$ and the min/max characterisation of the eigenvalue problem, then we know that the discrete eigenvalues λ_m^h will be approximating the continuous eigen value from above, i.e.

$$\lambda_i \leq \lambda_i^h \quad i = 1, \dots, m \quad \text{where } m \ll \dim(V_h)^*$$



ERN / GUERMOND FEM I

* This is not a functional analysis condition but it is obvious that if we want to approximate the first 10 eigenvalues to a decent degree of accuracy I'll have many more basis function.

PROPOSITION Let $\sigma_h^{(m)}$ be non-zero then we have, $\lambda_m \le \lambda_m^h \le (\sigma_h^{(m)})^{-2}\lambda_m$

PROOF We have already discussed one of the estimates.

Now we notice that $\ker(T_h) \cap \{u_1, \dots, u_m\} = \{0\}$ or else $\sigma_h^{(m)}$ is null. We then apply the RANK NULLITY THEOREM to see that $\operatorname{rk}(T_h|_{E_m}) = m$

Once again we consider the ℓ^2 projection from $T_h(E_m)$ onto $\{u_{h,1}, \dots, u_{h,m-1}\}$ and observe that the RANK NULLITY THEOREM implies $\{u_{h,1}, \dots, u_{h,m}\}^\perp \cap T_h(E_m)$ is non trivial, we call v_h such element and observe $v_h = \sum_{i \ge m}' v_i u_{h,i}$, hence as discussed above $R(v_h) \ge \lambda_m^h$.

$$\lambda_{h,m} \le R(v_h) \le \max_{w_h \in T_h(E_m)} \frac{a(w_h, w_h)}{m(w_h, w_h)} = \max_{v \in \{u_1, \dots, u_m\}} \frac{a(T_h v, T_h v)}{m(T_h v, T_h v)}$$

notice now that $a(T_h(v), T_h(v)) = a(v, T_h(v)) \le a(v,v)^{1/2} a(T_h v, T_h v)^{1/2}$ hence we have:

$a(T_h(v), T_h(v)) \le a(v,v)$, notice that here was important that a was POSITIVE DEFINITE.

$$\lambda_m^h \le \max_{v \in E_m} \frac{a(v,v)}{\|T_h(v)\|^2} \le \max_{v \in E_m} \frac{m(v,v)}{\|T_h(v)\|^2} \max_{v \in E_m} R(v) = (\sigma_h^{(m)})^{-2} \lambda_m.$$

to prove that $\max_{v \in E_m} R(v) = \lambda_m$ we used min/max characterisation.

LEMMA In the hypothesis of the previous proposition we have that:

$$\sigma_{hm}^2 \ge 1 - 2\sqrt{m} \frac{\|a\|}{\lambda_1} \max_{\substack{v \in E_m \\ m(v,v)=1}} \|v - T_h(v)\|_X^2$$

PROOF We begin observing that

$$\|T_h v\|_m^2 = \|v\|_m^2 - 2\, m(v, v - T_h(v)) + \|v - T_h v\|_m^2 \ge \|v\|_m - 2\, m(v, v - T_h v)$$

We now compute, $m(v, v - T_h v) = \sum_{i=1}^m v_i\, m(u_i, v - T_h v) = \sum_{i=1}^m \lambda_i^{-1} v_i\, a(u_i, v - T_h v)$

$$= \sum_{i=1}^m \lambda_i^{-1} v_i\, a(u_i - T_h u_i, v - T_h v)$$

GALERKIN ORTHOGONALITY

Hence we have,

$$m(v, v - T_h v) \le \frac{\|a\|}{\lambda_1} \|v - T_h v\|_X \sum_{i=1}^{m} |v_i|\, \|u_i - T_h u_i\|_X$$

$\lambda_1 \le \lambda_i$

This is bounded by estimates on H^1 projection.

$\sum_{i=1} |v_i|^2 = 1 \Rightarrow \sum_{i=1} |v_i| \le \sqrt{m}$

$$\le \frac{\|a\|}{\lambda_1} \|v - T_h v\|_X \sqrt{m}$$

THEOREM Let $m \in \mathbb{N} \setminus \{0\}$ then there exist h_0 such that $\forall h \le h_0$,

$$0 \le \lambda_m^h - \lambda_m \le C \max_{\substack{v \in E_m \\ m(v,v)=1}} \min_{v_h \in X_h} \|v - v_h\|_X^2$$

PROOF

From the previous lemma we know that:

$$\lambda^h_m - \lambda_m \le \left[(\sigma_h^{(m)})^{-2} - 1 \right] \lambda_m \le 2\sqrt{\lambda_m} \frac{\|a\|}{\lambda_1} \max_{\substack{v \in E_m \\ m(v,v)=1}} \|v - T_h v\|^2_X$$ by Céa's lemma we conclude.

Can we provide an a priori error bound also on the eigenfunction?

THEOREM Let $m \in \mathbb{N} / \{0\}$ and assume that λ_m is a SIMPLE eigenvalue, furthermore let us consider as in the previous proposition $h \in (0, h_0]$. Then we have the following a priori error estimate on the eigenfunctions,

$$\| u_m - u^h_m \|_X \le C \max_{\substack{v \in E_m \\ m(v,v)=1}} \min_{v_h \in X_h} \| v - v_h \|_V \qquad \text{(THM I)}$$

PROOF We begin using the coercivity of $a: X \times X \to \mathbb{R}$,

$$\alpha \| u_m - u^h_m \|^2_X \le \alpha\, a(u_m - u^h_m, u_m - u^h_m)$$

$$= \lambda_{hm} + \lambda_m - 2\lambda_m\, m(u_m, u^h_m)$$

we are using $m(u,u)=1$

$$= \lambda_{hm} + \lambda_m - \lambda_m \left[m(u_m, u_m) - m(u_m - u^h_m, u_m - u^h_m) + m(u^h_m, u^h_m) \right]$$

$$= \lambda_{hm} + \lambda_m - 2\lambda_m + \lambda_m \| u_m - u^h_m \|^2_m$$

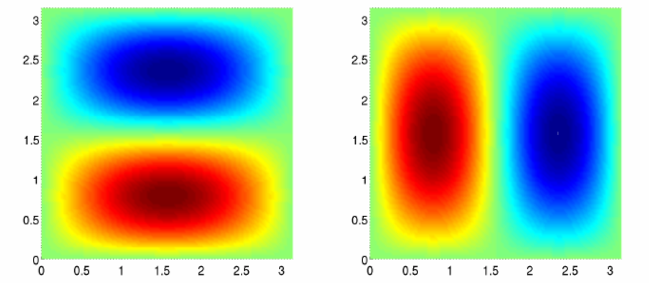
$$= \lambda_{hm} - \lambda_m + \lambda_m \| u_m - u^h_m \|^2_m$$

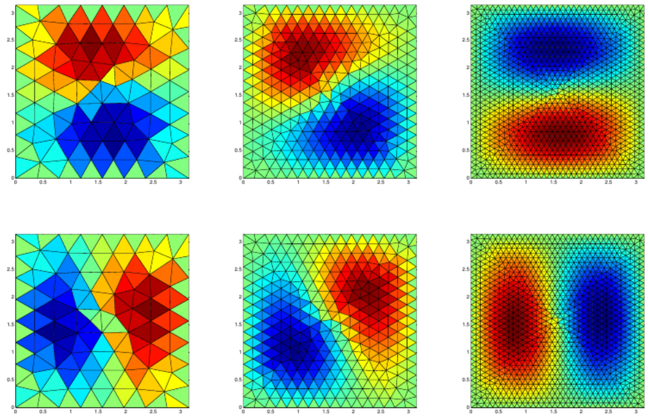
$$\le C \max_{\substack{v \in E_m \\ \|v\|=1}} \min \| v - v_h \|^2_X + \lambda_m \| m \| \, \| u_m - u^h_m \|^2_X$$

$$\le C \max_{\substack{v \in E_m \\ m(v,v)=1}} \min \| v - v_h \|^2_X + \lambda_m \| m \| \, \| u_m - T_h u_m \|_X$$

$$\underset{\text{APPLYING CEA'S LEMMA}}{\le} C(\|a\|, \|m\|) \max_{\substack{v \in E_m \\ m(v,v)=1}} \min \| v - v_h \|_X$$

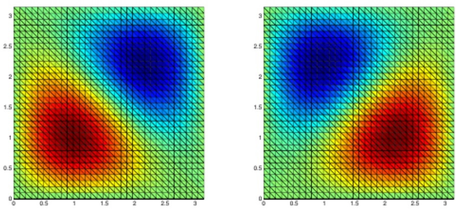
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
#	9	56	257	1106	4573



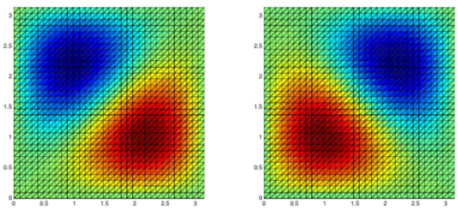


Uniform mesh

These are Type I meshes.



Uniform mesh (reversed)



REMARK What happens when we don't have a SIMPLE eigenvalue? We need to use BABUSKA-OSBORN theory, because we see phenomenon that looks like the one in the pictures.

FIGURES You can find this figures in BOFFI ACTA NUMERICA 2010.

REMARK The Poincaré inequality can be evaluated looking at it as an Ritz quotient, i.e.

$$C_P = \inf_{v\in X} \frac{\|\nabla v\|}{\|v\|} = \inf_{v\in X} \frac{a(v,v)}{(v,v)} = \lambda_1$$

where λ_1 is the first eigenvalue of the LAPLACIAN.

EXAMPLE (Stokes Eigenvalue problem) We can discretise the Stokes eigenvalue problem using divergence free element, i.e. we choose $X_h \subseteq \{ \underline{v} \in H^1_0 : \nabla \cdot v = 0 \}$. The Stokes problem in this setting looks like:

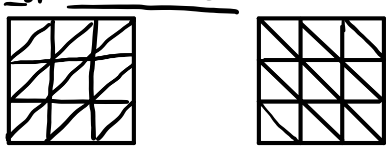
$$\text{find } u_h \in X_h : \quad (\varepsilon(u_h), \varepsilon(v_h))_{L^2} = \lambda (u_h, v_h)_{L^2} \quad \forall v_h \in X_h$$

* This is the primal formulation of STOKES.

hence we end up with the estimate,

$$0 \le \lambda^h_m - \lambda_m \le C \max_{\substack{v \in E_m \\ \|v\|=1}} \min_{v_h \in X_h} \|v - v_h\|^2_X$$

Type I Meshes



LEMMA Let $w \in H^s(\Omega)$ with $s \ge 2$, and assume that $\nabla \cdot w = 0$, then on type I mesh:

$$\inf_{w_h \in \mathbb{P}_p(\mathcal{T}_h)} \|w - w_h\|_{H^1(\Omega)} \le C \|w\|_{H^s} \begin{cases} h^{\min(p-1,s-1)} & p \in \{1,2,3\} \\ h^{\min(p,s-1)} p^{-(s-1)} & p \ge 4 \end{cases}$$

For this result check Charlie's "UNLOCK THE SECRETS OF LOCKING"

Scott-Vogelius element will approximate eigenvalues from above and converge with optimal rate on smooth domains and $p \geq 4$.