Primer on Bounded and Compact Operators in Banach and Hilbert spaces

Banach setting · Hilbert setting · Galerkin setting

The primary focus of this course will be the finite element approximation the spectra of compact operators in Banach spaces.

From this moment on we will denote $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two Banach spaces with the corresponding norm. Furthermore, we will denote $\mathcal{L}(X,Y)$ the set of linear operators from X to Y.

Definition Let $T \in \mathcal{L}(X,Y)$ we say that T is a bounded operator if it $\exists C_T > 0$ such that $\forall x \in X$ $\|Tx\|_Y \leq C_T \|x\|_X$. We will denote the set of all bounded linear operators $\mathcal{B}(X,Y)$. □

The space of bounded linear operators equiped with the norm, $\|T\|_{\mathcal{B}(X,Y)} := \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|}$ is a Banach space [Rudin].

Theorem Let $T \in \mathcal{B}(X,Y)$ be a bijective operator then $T^{-1} \in \mathcal{B}(Y,X)$. In particular $\|T^{-1}\|_{\mathcal{B}(Y,X)} \geq \|T\|^{-1}_{\mathcal{B}(X,Y)}$.

Proof This result is a consequence of the open mapping theorem see [Rudin]. □

Theorem (First Stability Estimate) Let $T \in \mathcal{B}(X,Y)$ be a bijective operator, then $\forall \Delta \in \mathcal{B}(X,Y)$ such that $\|\Delta\|_{\mathcal{B}(X,Y)} \leq \|T^{-1}\|^{-1}_{\mathcal{B}(Y,X)}$ then $T+\Delta$ is bijective, $(T+\Delta)^{-1} \in \mathcal{B}(Y,X)$ and the following bound holds,

$$\|(T+\Delta)^{-1}\|_{\mathcal{B}(Y,X)} \leq \|T^{-1}\|_{\mathcal{B}(Y,X)} \left[1 - \|\Delta\|_{\mathcal{B}(X,Y)} \|T^{-1}\|_{\mathcal{B}(Y,X)}\right]^{-1} \quad (*).$$

Proof $X \xrightarrow{\Delta} Y \xrightarrow{T^{-1}} X$

We begin observing that since $\|\Delta T^{-1}\|_{\mathcal{B}(X,X)} \leq \|\Delta\|_{\mathcal{B}(X,Y)} \|T^{-1}\|_{\mathcal{B}(Y,X)}$ we know that the Neumann series $\sum_{k=0}^{\infty} (-\Delta T^{-1})^k$ converges. In particular the Neumann series converges to $(I_X + \Delta T^{-1})^{-1}$, and we notice that $\|(I_X + \Delta T^{-1})^{-1}\|_{\mathcal{B}(X,X)} \leq \sum_{k=0}^{\infty} \|-(\Delta T)^k\|_{\mathcal{B}(X,X)} \overset{*}{=} \frac{1}{1 - \|\Delta T^{-1}\|_{\mathcal{B}(X,X)}}$.

* I've used $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r| < 1$.

Notice now that $T + \Delta = (I_X + \Delta T^{-1}) T$ hence since T and $(I_X + \Delta T^{-1})$ are both invertible also $T + \Delta$ is invertible, and in particular $(T+\Delta)^{-1} = T^{-1}(I_X + \Delta T^{-1})^{-1}$ hence $\|(T+\Delta)^{-1}\| \leq \|T^{-1}\|_{\mathcal{B}(Y,X)} \left(1 - \|\Delta\|_{\mathcal{B}(X,Y)} \|T^{-1}\|_{\mathcal{B}(Y,X)}\right)^{-1}$. □

It is now time to introduce the notion of adjoint of $T \in \mathcal{B}(X,Y)$.

From this moment on we will denote X^* the dual space to X, i.e. $\mathcal{B}(X,\mathbb{R})$ and we will write ${}_{X^*}\langle x_*, x \rangle$ to denote the action of the element $x_* \in \mathcal{B}(X,\mathbb{R})$ on the element $x \in X$. We now deal with the construction of an adjoint operator, let us consider $y_* \in Y^*$ we can then introduce the operator, $x \xmapsto{G_{y_*}} {}_{Y^*}\langle y_*, Tx \rangle_Y$. Notice that this operator belongs to X^*, by virtue of

$$\left| {}_{Y^*}\langle y_*, Tx \rangle_Y \right| \leq \|y_*\|_{\mathcal{B}(Y,\mathbb{R})} \|Tx\|_Y \leq \|T\|_{\mathcal{B}(X,Y)} \|y_*\|_{\mathcal{B}(X,\mathbb{R})} \|x\|_X .$$

Lemma (Riesz) Let $A \subsetneq X$ be closed wrt X, then $\forall \varepsilon > 0$ $\exists x \in X$ such that $\|x\|_X = 1$ and $\gamma(x, A) \geq 1 - \varepsilon$. [Brezis, Lem. 6.1]

Definition Given an operator $T \in \mathcal{B}(X,Y)$ we define its adjoint as the operator $T^* \in \mathcal{B}(Y^*, X^*)$ such that, $T^*: Y^* \to X^*$ $\quad y_* \mapsto G_{y_*} \in \mathcal{B}(X,\mathbb{R})$, where G_{y_*} is defined as above.

If $(X, \|\cdot\|_X)$ is a Hilbert space many more things can be said about such operators. To make the exposition more clear we will denote $(H, \|\cdot\|_H)$ and $(K, \|\cdot\|_K)$ two Hilbert spaces and $(\cdot,\cdot)_H$, $(\cdot,\cdot)_K$ the respective inner product. We will make heavy use of Riesz-Fréchet identification, i.e.

Theorem (Riesz-Frechet identification) Given any $h_* \in H^*$ there exist a unique $h \in H$ such that ${}_{H^*}\langle h_*, k \rangle_H = (h,k)_H$ for any $k \in H$. Furthermore $\|h\|_H = \|h_*\|_{H^*}$. We call the Riesz map of H, $R_H: H^* \to H$ $\quad h^* \mapsto h$. R_H is bijective.

Proof The proof of this result can be found in [Brezis, Theorem 5.5]

The adjoint operator in the Hilbert setting can be defined starting from the Banach adjoint as follows

Definition Given $T \in \mathcal{B}(H,K)$ and its Banach adjoint $T^* \in \mathcal{B}(K^*, H^*)$ the we define the Hilbert adjoint of T, the operator $T': K \to H$ $\quad k \mapsto R_H T^* R_K^{-1} k$.

Proposition Let $T \in \mathcal{B}(H,K)$, then the Hilbert adjoint T' can also be defined the operator $T' \in \mathcal{B}(K,Y)$ such that $\forall (h,k) \in H \times K$ we have $(Th, k)_K = (h, T'k)_H$ $(*)$.

Proof We compute $(h, T'k)_H = (h, R_H T^* R_K^{-1} k)_H = {}_{H^*}\langle T^* R_K^{-1} k, h \rangle_H = {}_{H^*}\langle G_{R_K^{-1} k}, h \rangle_H$

Where $G_{R_K^{-1} k}: H \to \mathbb{R}$ $\quad h \mapsto {}_{K^*}\langle R_K^{-1} k, Th \rangle$, hence by using the Riesz rappresentation we have that ${}_{H^*}\langle G_{R_K^{-1} k}, h \rangle_H = (k, Th)_K$, thus we have the identity $(Th, k)_K = (h, T'k)_H$.

We can now define two macro category of ENDOMORPHISMS, which will see later have particular spectral properties.

DEFINITION We say that an operator $T \in \mathcal{B}(H,H)$ is self-adjoint if $T' = T$. While we say that T is normal if $T'T = TT'$.

REMARK It is important to notice that in the previous definition T was an ENDOMORPHISM, i.e. $T \in B(H,H)$.

REMARK We have here omitted to identify H with H^* via the Riesz map to avoid unpleasant paradoxical situation such as the fact that if $H_0^1 \simeq H^{-1}$ and $L^2 \simeq (L^2)^*$ we have $H_0^1 \subseteq L^2 \simeq (L^2)^* \subseteq H^{-1} \simeq H_0^1$ thus meaning that $L^2 = H_0^1$

THEOREM If $T \in \mathcal{B}(H,H)$ is self-adjoint also $T^{-1} \in B(H,H)$ is self-adjoint.

DEFINITION We say an operator $T \in \mathcal{L}(X,Y)$ is compact if for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ we can extract a converging subsequence from $\{Tx_n\}_{n\in\mathbb{N}} \subseteq Y$. We will denote the space of compact operators from X to Y as $K(X,Y)$.

COROLLARY All compact operators are bounded operators, i.e. $K(X,Y) \subseteq B(X,Y)$.

PROOF Lets pick $T \in K(X,Y)$ and assume that $T \notin \mathcal{B}(X,Y)$, then it must exist a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\|Tx_n\|_Y > n$, $\forall n \in \mathbb{N}$. Such sequence has no bounded subsequence and therefore also no Cauchy subsequences, implying $T \notin K(X,Y)$.

PROPOSITION Given $S \in B(X,X)$ and $T \in K(X,Y)$ then $TS \in K(X,Y)$.

PROOF Bounded operators preserves the boundedness of a sequences, hence TS must be compact.

THEOREM (Fredholm alternative) Let $T \in K(X,X)$ then $N(T-I) = \{0\}$ if and only if $R(T-I) = X$.
Furthermore $N(T-I)$ is finite-dimensional.

PROOF

COROLLARY Let $T \in K(X,X)$ and $z \in \mathbb{C}\setminus\{0\}$ then either there is non-trivial solution to the equation $Tx = zx$ or $T - zI_X$ is bijective and $(T-zI)^{-1} \in \mathcal{B}(X,X)$.

PROOF Notice that if $T \in K(X,X)$ then $z^{-1}T \in K(X,X)$ for any $z \neq 0$. We can apply the previous theorem to state that $N(T-zI_X) = \{0\}$ if and only if $R(T-zI) = X$, hence either the operator is bijective and has bounded inverse because $T \in \mathcal{B}(X,X)$ and we can apply the bounded inverse theorem to $(T-zI)$.

EXAMPLE Let us consider the differential operator defined as follows, $T_0: W_0^{1,1+\varepsilon}([a,b],\mathbb{R}) \to L_0^{1+\varepsilon}([a,b],\mathbb{R})$, defined by $T_0 u := u'(x)$. We can compute the Banach adjoint $T_0^*: L^q([a,b],\mathbb{R}) \to W^{-1,q}([a,b],\mathbb{R})$, with $q = \varepsilon^{-1}(1+\varepsilon)$. Following the above definition of Banach adjoint we have that can be defined by its action $\forall f^* \in L^q([a,b])$, i.e. ${}_{W^{-1,q}}\langle T^*_{f_*}, u\rangle_{W^{1,q}} = {}_{L^q}\langle f_*, Tu\rangle_{L^p}$. Unfortunately this is as far as the definition of Banach adjoint.

One might think the situation is different for the operator $T_0: H_0^1(\Omega) \to L_0^2(\Omega)$, yet we can see trying to apply (*) that no explicit computation can be applied to obtain an expression for T'.
$(Tu, f)_{L^2} = \int u'f\,dx = -\int uf'\,dx \neq (u, T')_{H_0^1(\Omega)}$

EXAMPLE let us now consider the the operator $T_1: \overbrace{W_0^{1,1+\varepsilon}([a,b])}^{X} \to \overbrace{W^{-1,1+\varepsilon}([a,b])}^{Y}$ $u \mapsto -u''$, once again not much can be said of T_1 neither in the Banach setting nor in Hilbert setting except from the fact that $T_1 \in B(X,Y)$.

Since $T_1 \in B(X,Y)$ we play the "trick of the inverse". In fact we will consider the solution operator $T_1^{-1}: Y \to X$ and consider such operator as $T_1^{-1}: L^{1+\varepsilon}([a,b]) \to L^{1+\varepsilon}([a,b])$. We then notice that if $-u'' = f$ then $-\int u''u = \int u'u' = \int fu$ hence using Poincare inequality and Hölder inequality we get the following:

$\|u'\|^2_{L^2} \leq \|u\|_{L^q} \|f\|_{L^{1+\varepsilon}}$ now notice that by the embedding $W^{1,1+\varepsilon}([a,b]) \subseteq L^q([a,b])$ applying $\|u'\|_{L^{1+\varepsilon}} \leq \|u\|_{L^2}$ and the Poincare inequality in the Sobolev space $W^{1,1+\varepsilon}([a,b])$ we o-ach that for any $f \in L^{1+\varepsilon}([a,b])$ we have $u \in W_0^{1,1+\varepsilon}([a,b])$ and since $W^{1,1+\varepsilon}([a,b])$ is compactly embedded in $L^{1+\varepsilon}([a,b])$ we see that $T_1^{-1} : L^{1+\varepsilon}([a,b]) \longrightarrow L^{1+\varepsilon}([a,b])$ is compact.

In the Hilbert setting we can study the inverse of $T_1 : H_0^2([a,b]) \to L^2([a,b])$ with extended domain $T_1^{-1} : L^2([a,b]) \to L^2([a,b])$ and apply the same reasoning to see that T_1^{-1} is compact. Furthermore let us denote $u = T_1^{-1} f$ and $v = T_1^{-1} g$ then we have that $-u'' = f$ and $-v'' = g$, thus multiplying by v and u respectively and integrating we get $\forall f, g \in L^2([a,b])$: $(f, v) = \int_a^b -u''v = \int_a^b u'v' = -\int_a^b u v'' = (u, g) \Rightarrow (T_1^{-1} g, v)$ is equal to $(u, T_1^{-1} f)$, hence T_1^{-1} is SELF-ADJOINT. By elliptic regularity we know that $R(T_1^{-1}) = H_0^2([a,b])$ hence $(T_1^{-1})^{-1} = T_1$ and thus also T_1 is SELF-ADJOINT.

Let us consider now a Galerkin discretisation of the PDE, $\mathcal{L}u := -u'' + u'$ i.e. we fix a discrete subspace H_n such that $\dim X_n = n$ and construct the sequence of discrete operators $\{T_n\}_{n \in \mathbb{N}}$ $T_n : H_n \to H_n$ $(T_n)_{ij} = (\mathcal{L}\phi_i, \phi_j)$, where $\{\phi_i\}_{i=1}^n$ forms a basis of H_n. In particular, we will focus our attention on the CONFORMING setting, i.e. $H_n \subseteq H$ for any $n \in \mathbb{N}$, and we will ask that for any $x \in H$ the operation $\mathcal{L}x$ makes sense at least from a distributional point of view.

DEFINITION Given two differential equation here rappresented by their action $\mathcal{L}h$ and $\hat{\mathcal{L}}\hat{h}$, we say that they are one the FORMAL L^2 ADJOINT on H of one another if $(\mathcal{L}h, \hat{h})_{L^2} = (h, \hat{\mathcal{L}}\hat{h})_{L^2}$ for any $\hat{h} \in H$.

We observe that the sequence of operators $\{T_n^T\}_{n \in \mathbb{N}}$ can be defined in terms of the formal adjoint, in fact $(T_n)^T_{ij} = (\mathcal{L}\phi_j, \phi_i) = (\hat{\mathcal{L}}\phi_i, \phi_j)$. Hence given a sequence of Galerkin discretisation of $\mathcal{L}$, $\{T_n\}_{n \in \mathbb{N}}$, their transposes $\{T_n^T\}_{n \in \mathbb{N}}$ are a sequence of Galerkin discretisation of the formal L^2 adjoint on H of $\mathcal{L}$.

We already discussed the difference between the BANACH and HILBERT adjoints, in the following example we clarify the difference between the formal adjoint and the Hilbert adjoint.

EXAMPLE Let us consider the ODE $\mathcal{L}h = -h''$ together with the boundary conditions $h(a) = h(b) = 0$. Integrating by parts we can compute $(\mathcal{L}h, \hat{h})$, i.e.

$$(\mathcal{L}h, \hat{h})_{L^2} = \int_a^b -h''\hat{h} + h'\hat{h} = \int_a^b h'\hat{h}' - \underbrace{\Big|_a^b h'\hat{h}}_{\substack{\text{This vanish because} \\ \hat{h} \in H \text{ thus } \hat{h}(a) = \hat{h}(b) = 0}} = -\int_a^b h\hat{h}'' - \underbrace{\Big|_a^b h\hat{h}'}_{\substack{\text{This vanish because} \\ h \in H.}} = -\int_a^b h\hat{h}''.$$

This suggest that the $\mathcal{L}$ is the formal L^2 adjoint on H of it-self, i.e. $\hat{\mathcal{L}} = \mathcal{L}$. At first this observation agrees with the fact that the solution operator we disused in a previous remark is self-adjoint. Yet we need to proceed with caution in fact for an operator to be self-adjoint it needs to be an ENDOMORPHISM for this reason we decide to consider the operator $T : L^2([a,b]) \to L^2([a,b])$, with $\mathrm{dom}(T) = H_0^2([a,b])$. We are now in the setting of UNBOUNDED OPERATORS which is well beyond the topic of this notes, so we will only make some comments. While the reader might think T is the operator associated with $\mathcal{L}$ and T' is the operator associated with $\hat{\mathcal{L}}$ thus $\mathcal{L} = \hat{\mathcal{L}}$ implies T is self-adjoint this is not really the case. In fact while $\mathcal{L} = \hat{\mathcal{L}}$ implies the symmetry of the operator T we have no guarantee that $D(T) = D(T^*)$, never the less in the particular case of the Laplacian it is possible to construct a SELF-ADJOINT extension of T [KATO]. Not only T' is not the is not the operator associated to T' but more over given a Galerkin discretisation $\{T_n\} : H \to H$ is not true that $\{T_n^T\}_{n \in \mathbb{N}}$ is a Galerkin discretisation of T'. If we consider as operator the one associated with the differential equation $\mathcal{L}h = h'$ with boundary conditions $h(a) = 0$, i.e. $T : L^2([a,b]) \to L^2([a,b])$ with $\mathrm{dom}(T) = \{h \in H^2([a,b]) : h(a) = 0\}$, then $\hat{\mathcal{L}}h = -h'$ which shows that a the transpose of a Galerkin discretisation is not discretising the adjoint operator since the adjoint operator in this case $\mathrm{dom}(T')$

$= \{ h \in H^2([a,b]) : h(a) = 0 \}$ [KATO, Ex III.2.6]. Similar issue arise when discussing normality, let us consider the operator associated with the differential equation $\ell h = h'$ with boundary conditions $h(a) = h(b) = 0$, i.e. $T : L^2([a,b]) \to L^2([a,b])$ with $\mathrm{dom}(T) = H^2_0([a,b])$. While one might think that the operator T is NORMAL because $\ell \hat{\ell} = -u'' = \hat{\ell} \ell$, this is not the case. In fact $\mathrm{dom}(T^*) = H^2([a,b])$ and thus $\mathrm{dom}(TT^*) \neq \mathrm{dom}(T^*T)$ proving that T is not a NORMAL OPERATOR [KATO EXAMPLE V.3.25].

* MISSING A DISCUSSION ON FINITE RANK OPERATORS.