SPECTRAL APPROXIMATION BY FINITE RANK OPERATORS

In this section we focus on the approximation of compact operator. The FRAMEWORK here presented is the due to BABUSKA-BRAMBLE-OSBORN. In [BRAMBLE-OSBORN 1973] a theory for the approximation of NON SELF-ADJOINT COMPACT operators, in Hilbert setting was presented under the hypothesis that the sequence of finite rank approximation converged UNIFORMLY. In [BRAMBLE 1975] the theory was extended to the Banach case and under the hypothesis that the finite-rank sequence of approximation was only point-wise convergent. In [BOFFI-BREZZI-GASTALDI 2000] a counter example to the the fact that the sequence of finite-rank approximator is point-wise convergent suffices to guarantee the approximation of the SPECTRUM with no SPURIOUS MODES was found in the context of MIXED FEM. In [BABUSKA-OSBORN 1989] more refined estimates for the rate of convergence of the spectrum then the one presented in [OSBORN 1975] were in the the Hilbert setting for SELF-ADJOINT operators.

Given $T \in K(X,X)$ we will here consider a sequence of $\{T_n\}_{n\in\mathbb{N}} \subseteq K(X,X)$ with $\dim R(T_n) < +\infty$ for any $n \in \mathbb{N}$ and such that $\lim_{n\to\infty} \|T-T_n\|_{\mathcal{B}(X,X)} = 0$. Under this hypothesis the sequence $\{T_n\}_{n\in\mathbb{N}}$ is COLLECTIVELY COMPACT, i.e. the set $\{T_n x : \|x\|=1, \text{ with } n \in \mathbb{N}\}$ is sequentially compact. In fact, let us consider a sequence $\{T_{n_k} x_{n_k}\}$ and prove that we can extract a Cauchy subsequence: $\|T_{n_k} x_{n_k} - T_{n_j} x_{n_j}\| \leq \| T_{n_k} x_{n_k} - T x_{n_k} + T x_{n_k} - T x_{n_j} - T_{n_j} x_{n_j} + T x_{n_j} \|$

$$\leq \|T_{n_k} x_{n_k} - T x_{n_k}\| + \|T x_{n_k} - T x_{n_j}\| + \|T x_{n_j} - T_{n_j} x_{n_j}\|$$

We now extract a converging subsequence from $\{T x_{n_k}\}$ here by an abuse of notation still denoted $\{T x_{n_k}\}$, which can be done because T is compact. We can then conclude the proof observing that all the term on the right hand side vanish as $k \to \infty$.

Let us consider $\mu \in \sigma_T \setminus \{0\}$ and denote Γ a loop wrapping once around μ, and not any other element of σ_T.

LEMMA (Approximability of the PSEUDO SPECTRA) Let T and $\{T_n\}_{n\in\mathbb{N}}$ be as in the above setting, for any fixed $\varepsilon > 0$ and $\delta < \varepsilon < \delta'$ it $\exists\, \ell^* \in \mathbb{N}$ such that $\sigma_{T_n,\delta} \subseteq \sigma_{T,\varepsilon}$ $\forall n > \ell^*$.

PROOF We notice that we want to prove $\sigma_{T_n,\delta} \subseteq \sigma_{T,\varepsilon} \iff (\sigma_{T,\varepsilon})^c \subseteq (\sigma_{T_n,\delta})^c$.

Hence we assume that $\|R_T(z)\| \leq \frac{1}{\varepsilon}$ and we want to prove that $\|R_{T_n}(z)\| \leq 1/\delta$.

We begin rewriting $T_n - zI = T - zI + T_n - T = A + \Delta$ with $A = T - zI$ and $\Delta = T - T_n$. Notice now that since $T_n \xrightarrow{u} T$ we know $\|\Delta\|$ is arbitrarly small hence we can apply the "FIRST STABILITY ESTIMATE" since we can pick sufficently large so that $\|\Delta\| \leq \|A^{-1}\|^{-1} = \|R_T(z)\|^{-1}$. Hence we know that $R_{T_n}(z)$ is well defined and:

$$\|R_{T_n}(z)\| = \|(A+\Delta)^{-1}\| \leq \frac{\|\overbrace{R_T(z)}^{A^{-1}}\|}{1 - \|\underbrace{T-T_n}_{\Delta}\| \|\underbrace{R_T(z)}_{A^{-1}}\|}$$

We can now pick n sufficiently large so that $1 - \|T-T_n\| \|R_T(z)\| \geq \frac{\delta}{\varepsilon}$ (notice $\delta < \varepsilon$).

Then in this case we know that $(1 - \|T-T_n\| \|R_T(z)\|)^{-1} \leq \frac{\varepsilon}{\delta}$ so we have:

$\|R_{T_n}(z)\| = \|(A+\Delta)^{-1}\| \leq 1/\varepsilon \; \varepsilon/\delta \leq 1/\delta$.

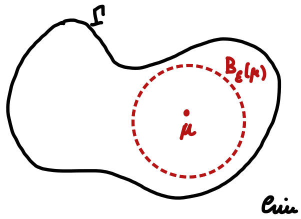
LEMMA (*) Let $A, B, C \in K(X,X)$ such that $(A-B)$ and $(C-B)$ are invertible then,

$$(A-B)^{-1}(A-C)(C-B)^{-1} = -\left((A-B)^{-1} - (C-B)^{-1}\right). \quad (*)$$

PROOF We begin observing that $A - C = (A-B) - (C-B)$. We notice that the LHS of (*) by virtue of the previous identity as $(A-B)^{-1}[(A-B)-(C-B)](C-B)^{-1} = (A-B)^{-1}(A-B)(C-B)^{-1} - (A-B)(C-B)(C-B)^{-1}$ hence we get $(C-B)^{-1} - (A-B)^{-1}$ it remains to flip the answer to conclude.

COROLLARY In the setting described above fixed $\mu \in \sigma_T \setminus \{0\}$ we consider a loop $\Gamma \subseteq \mathbb{C}$ wrapping around μ and no other element of σ_T. Then it $\exists\, n^* \in \mathbb{N}$ such that $\forall n > n^*$ $\Gamma \subseteq \rho(T_n)$.

PROOF Let us pick $\varepsilon > 0$ as in the figure below. From the bound $\|R_T(z)\| > \gamma(z, \sigma_T)^{-1}$ we then know that $\Gamma \subseteq (\sigma_{T,\varepsilon})^c$ and from the previous result you see that it $\exists\, \ell^* \in \mathbb{N}$ such that $\Gamma \subseteq (\sigma_{T_n,\delta})^c \subseteq \rho_{T_n}$.



This above result tells us that we can def. the projector $\Lambda_{T_n,\Gamma_\mu} : X \longrightarrow X$ for all $n > \ell^*$. Furthermore notice that if T_n converges uniformly to T we have that $(T_n - zI)$ converges uniformly to $T - zI$, and thus $\lim_{n\to\infty} \|R_T(z) - R_{T_n}(z)\| = 0 \quad \forall z \in \rho_T \cap \rho_{T_n}$. Using the linearity of the integral (or if one prefers a more rigorous argument passing to the limit the Cauchy sequences discretising the integral) we get:

$$\|\Lambda_{T,\Gamma_\mu} x - \Lambda_{T_n,\Gamma_\mu} x\| = \left\| \frac{1}{2\pi i} \int_\Gamma (R_T(z) - R_{T_n}(z))\, x \right\| \overset{(*)}{=} \frac{1}{2\pi} \left\| \int_\Gamma R_{T_n}(z)(T - T_n) R_T(z)\, x\, dz \right\|$$
$$\leq \frac{1}{2\pi} |\Gamma| \sup_{z\in\Gamma} \|R_{T_n}(z)\| \sup_{z\in\Gamma} \|R_T(z)\| \, \|(T - T_n)\|$$

notice that since $\{T_n\}$ is collectively compac- we have that both sups are bounded in n, while $\lim_{n\to\infty} \|T - T_n\| = 0$, hence $\lim_{n\to\infty} \|\Lambda_{T,\Gamma_\mu} - \Lambda_{T_n,\Gamma_\mu}\| = 0$, hence we know that Λ_{T_n,Γ_μ} converges uniformly to Λ_{T,Γ_μ}.

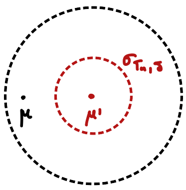
PROPOSITION $\dim R(\Lambda_{T,\Gamma_\mu}) = \dim R(\Lambda_{T_n,\Gamma_\mu})$ for any $n > \ell^*$ starting from a certain ℓ^*.

PROOF Since T and T_n are compact operators by FREDHOLM ALTERNATIVE THEOREM, we know that the eigen space are finite dimensional. Since T_n converges uniformly to T_n we know that $\dim R(\Lambda_{T_n,\Gamma_\mu}) \to m := \dim R(\Lambda_{T_n,\Gamma_\mu})$. Notice that because the sequence converges to a constant number it $\exists\, \ell^*$ such that $\forall n > \ell^*$ $\dim R(\Lambda_{T_n,\Gamma_\mu}) = m$.

COROLLARY Let T and $\{T_n\}_{n\in\mathbb{N}}$ be as in the setting of this section. Then fixed $\mu \in \sigma_T \setminus \{0\}$, with ASCENT $\alpha \in \mathbb{N}$, by the above proposition there $\exists\, \ell^*$ such that $\forall n > \ell^*$ there exist $m = \dim N((T - \mu I)^\alpha)$ eigenvalues of T_n, here denoted $\mu_n^{(j)}$ for $j = 1, \dots, m$, such that the $\lim_{n\to\infty} \mu_n^{(j)} = \mu$.

PROOF From the above discussion and proposition we know that fixed a curve Γ_μ as above it $\exists\, \ell^*$ s.t. Λ_{T_n,Γ_μ} is well-defined and $\dim R(\Lambda_{T,\Gamma_\mu}) = \dim R(\Lambda_{T_n,\mu})$. Then since $\Lambda_{T_n,\mu}$ is the projection onto the eigen space of T_n we know that there are $\dim R(\Lambda_{T,\mu})$ eigenvectors of T_n associated to the eigen-values $\mu_n^{(j)}$ for $j = 1, \dots, \dim R(\Lambda_{T,\Gamma_\mu})$. Since Λ_{T,Γ_μ} is the projection onto the eigenvectors and Γ only wraps around μ and no other element of σ_T we have that $m = \dim R(\Lambda_{T,\Gamma_\mu})$. To conclude we consider a sequence of Γ shrinking to μ to prove that $\lim_{n\to\infty} \mu_n^{(j)} = \mu$ for any $j = 1, \dots, m$.

Notice that the "Approximability of the PSEUDO SPECTRA" guarantees the absence of SPURIOUS EIGEN VALUES. In fact, let us consider a sequence of operator $\{T_n\}_{n\in\mathbb{N}}$ approximating the spectrum of of a continuous operator T, i.e. we assumed that fixed any $\mu \in \sigma_T \setminus \{0\}$, with ASCENT α it $\exists\, \ell^*$ such that $\forall n > \ell^*$ there exist $m = \dim N((T - \mu I)^\alpha)$ eigenvalues of T_n, here denoted $\mu_n^{(j)}$ for $j = 1, \dots, m$ and $\lim_{n\to\infty} \mu_n^{(j)} = \mu$. If this is the case a possible adverse event is that it in addition to the eigenvalues converging to μ the discrete sequence of operators T_n would present an extra sequence of eigenvalues converging to $\mu' \notin \sigma_T \setminus \{0\}$. For any $\varepsilon > 0$ and we notice that it $\exists\, \ell^*$ such that $\forall n > \ell^*$ we have $\mu' \in \sigma_{T_n,\delta} \subset \sigma_{T,\varepsilon}$. Letting $\varepsilon \searrow 0$ we should find that we μ' is arbitrarly close to μ, which is a contradiction



Thus we notice that if T_n converges uniformly to T, the spectrum of T_n converg-es to the spectrum of T with out any SPURIOUS eigenvalue and since there are is a number of discrete eigenvalues approximating $\mu \in \sigma_T \setminus \{0\}$ equal to the algebraic multipli-city of μ, with no SPURIOUS GENERALISED EIGENFUNCTION EITHER.

As we previously remarked if $T \in K(H,H)$ is SELF-ADJOIN the algebraic and geometric multiplicity are the same which allow us to conclude that in this case if $\{T_n\}_{n\in\mathbb{N}}$ is a sequence converging uniformly to T, then $\{T_n\}_{n\in\mathbb{N}}$ approximates the spectrum of T with no SPURIOUS MODES.

We now address the question of how fast does the discrete spectrum converge. We will do so in terms of gap between the discrete and continuous eigenspaces.

DEFINITION Given two closed subspaces A and B of the Banach space X we define the gap between A and B as $\delta(A,B) = \sup_{\substack{x\in A\\ \|x\|=1}} \gamma(x,A)$. Notice that the notion of gap is not symmetric, i.e. in general $\delta(A,B)\neq\delta(B,A)$. We define the SYMMETRISED GAP as $\hat{\delta} = \max\{\delta(A,B), \delta(B,A)\}$. Many authors would refer to $\hat{\delta}(A,B)$ simply as the gap.

From the definition of gap we see that $\delta(A,B)\in[0,1]$ as does $\hat{\delta}(A,B)$. Furthermore $\delta(A,B)=0$ if and only if $M\subseteq N$, while $\hat{\delta}(M,N)=0$ if and only if $A=B$.

LEMMA Let A and B be two close sub-spaces of X, such that $\dim A = \dim B < +\infty$, then we have: $\delta(B,A)\le\delta(A,B)[1-\delta(A,B)]^{-1}$.

PROOF Let us proceed by contradiction, if $\delta(B,A)\ge\delta(A,B)[1-\delta(A,B)]^{-1}$ hence $\delta(A,B)\delta(B,A)^{-1}\le 1-\delta(A,B)$.
If $A=B$, then the results follows trivially, same if $\delta(A,B)=1$, hence we can assume $\delta(A,B)\in(0,1)$.
Hence either $A\subsetneq B$ or $B\subsetneq A$, but this is a contradiction since $\dim(A)=\dim(B)<+\infty$.

LEMMA Let A, B_1, B_2 be two closed subspaces of X, such that $X = B_1\oplus A = B_2\oplus A$ and let Π_1 and Π_2 be the projections of X along A onto B_1 and B_2 respectively. If $\|I-\Pi_2\|\delta(B_1,B_2)<1$, then we have:

$$\|\Pi_1-\Pi_2\|\le\left(\|\Pi_2\|\|I-\Pi_2\|\delta(B_1,B_2)\right)\left(1-\|I-\Pi_2\|\delta(B_1,B_2)\right)^{-1}.$$

PROOF This proof comes from [OSBORN 1975], and is due to C. LAY. Let $d=\sup_{\substack{x\in B_1\\ \|x\|=1}}\|x-Q_2x\|$ and let $\delta>\delta(B_1,B_2)$. If we pick $x\in B_1$ such that $\|x\|=1$, by the definition of $\delta(B_1,B_2)$ there exists $x'\in B_2$ such that $\|x-x'\|<\delta$. Since $(I-\Pi_2)\underset{\in B_2}{x'}=0$, we have $\|x-\Pi_2x\|=\|(I-\Pi_2)(x-x')\|\le\|I-\Pi_2\|\delta$. Now notice that this implies $d\le\|I-\Pi_2\|\delta$ for any $\delta>\delta(B_1,B_2)$, and therefore $d\le\|I-\Pi_2\|\delta(B_1,B_2)$ $(\circledast)$. We now observe that $R(I-\Pi_1)=N(\Pi_2)$, since $X=B_1\oplus A=B_2\oplus A$, implies $\Pi_2=\Pi_2\Pi_1$ and thus $\forall x\in X$ we have:

$$\|\Pi_1x\|\le\|\Pi_1x-\Pi_2\Pi_1x\|+\|\Pi_2\Pi_1x\|\le d\|\Pi_1x\|+\|\Pi_2x\| \Rightarrow \|\Pi_1x\|\le(1-d)^{-1}\|\Pi_2x\|.$$ Using $(\circledast)$ we get

$$\|\Pi_1x-\Pi_2x\|\le\|\Pi_1x-\Pi_2\Pi_1x\|\le d\|Q_1x\|\le\left(\|I-\Pi_2\|\delta(B_1,B_2)\|\Pi_2\|\right)\left(1-\|I-\Pi_2\|\delta(B_1,B_2)\right)^{-1}.$$

THEOREM There $\exists C_1>0$ and $\ell^*\in\mathbb{N}$ such that $\forall n>\ell^*$ we have $\hat{\delta}(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))\le C_1\|(T-T_n)|_{R(\Lambda_{T,\Gamma_\mu})}\|$.

PROOF We begin observing that $\forall x\in R(\Lambda_{T,\Gamma_\mu})$ with $\|x\|=1$, then we have $\|x-\Lambda_{T_n,\Gamma_\mu}x\|$ can be expressed as $\|(\Lambda_{T,\Gamma_\mu}-\Lambda_{T_n,\Gamma_\mu})\Lambda_{T,\Gamma_\mu}x\|\le\|(\Lambda_{T,\Gamma_\mu}-\Lambda_{T_n,\Gamma_\mu})\Lambda_{T,\Gamma_\mu}\|$, this because $\Lambda_{T,\Gamma_\mu}\Lambda_{T,\Gamma_\mu}=\Lambda_{T,\Gamma_\mu}$ and since $x\in R(\Lambda_{T,\Gamma_\mu})$ then $x=\Lambda_{T,\Gamma_\mu}x$. Thus we have $\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))\le\|(\Lambda_{T_n,\Gamma_\mu}-\Lambda_{T,\Gamma_\mu})\Lambda_{T,\Gamma_\mu}\|$.
Since we have proven that Λ_{T_n,Γ_μ} converges uniformly to Λ_{T,Γ_μ} then $\lim_{n\to\infty}\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))=0$.
We now apply one of the previous LEMMA to obtain that,

$$\delta(R(\Lambda_{T_n,\Gamma_\mu}),R(\Lambda_{T,\Gamma_\mu}))\le\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))\left[1-\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))\right]^{-1}.$$

Since we know that $\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))\searrow_0$ this means that it $\exists C_1>0$ such that: $\delta(R(\Lambda_{T_n,\Gamma_\mu}),R(\Lambda_{T,\Gamma_\mu}))\le C_1\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))$.
We have thus obtained $\hat{\delta}(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))\le(1+C_1)\delta(R(\Lambda_{T,\Gamma_\mu}),R(\Lambda_{T_n,\Gamma_\mu}))$, this estimate is more useful then the original one because we are only able to control the expression on the right hand side.
Now for $x\in R(\Lambda_{T,\mu})$ we have,

$$\|x-\Lambda_{T_n,\mu}x\|\le\|\Lambda_{T,\Gamma_\mu}x-\Lambda_{T_n,\Gamma_\mu}x\|=\left\|\frac{1}{2\pi i}\int_\Gamma(R_T(z)-R_{T_n}(z))x\right\|=\frac{1}{2\pi}\left\|\int_\Gamma R_{T_n}(z)(T-T_n)R_T(z)x\,dz\right\|$$

as previously discussed above the RHS can be simplified to be $C_1\|(T-T_n)|_{R(\Lambda_{T,\Gamma_\mu})}$, hence we obtained the desired result.

THEOREM There exist an $\ell^*\in\mathbb{N}$ and a constant $C>0$ such that $\forall n>\ell^*$ the following estimate holds, $|\mu-\frac{1}{m}\sum_{j=1}^{m}\mu^j_{(n)}|\le C_2\|(T-T_n)|_{R(\Lambda_{T,\Gamma_\mu})}\|$.

PROOF We begin observing that for a large enough n the discrete projection operator $\Lambda_{T_n,\Gamma_\mu} : R(\Lambda_{T,\Gamma_\mu}) \longrightarrow R(\Lambda_{T_n,\Gamma_\mu})$. This is because we know that Λ_{T_n,Γ_μ} converges uniformly to Λ_{T,Γ_μ} and $\Lambda_{T_n} x = 0$ implies $\|x\| = \|\Lambda_{T,\Gamma_\mu} x - \Lambda_{T_n,\Gamma_\mu} x\| \leq \|(\Lambda_{T,\Gamma_\mu} - \Lambda_{T_n,\Gamma_\mu})\Lambda_{T,\Gamma_\mu}\|$. Furthermore as we already proved since $\dim R(\Lambda_{T_n,\Gamma_\mu}) = \dim(\Lambda_{T,\Gamma_\mu}) = m < +\infty$ the operator is also surjective. Thus we can define the operator $\left(\Lambda_{T_n,\Gamma_\mu}\big|_{R(\Lambda_{T,\Gamma_\mu})}\right)^{-1} : R(\Lambda_{T_n,\Gamma_\mu}) \to R(\Lambda_{T,\Gamma_\mu})$. To shorten the notation from now on we will simply write Λ_n and Λ_n^{-1} to denote the operator Λ_{T_n,Γ_μ} restricted to $R(\Lambda_{T\,\Gamma_\mu})$ and its inverse. We notice that for a sufficiently large n, if $x \in R(\Lambda)$ and $\|x\| = 1$ we have that: $1 - \|\Lambda_n x\| = \|\Lambda x\| - \|\Lambda x\| = \|(\Lambda - \Lambda_n)\Lambda\| \leq 1/2$. Hence $\|\Lambda_n x\| \geq \frac{1}{2}$ which implies that $\|\Lambda_n^{-1}\| \leq \frac{1}{2}$ and thus for a sufficiently large n we have that $\Lambda_n \Lambda^{-1}$ is the identity on $R(\Lambda_n)$ and $\Lambda_n^{-1}\Lambda$ is the identity on $R(\Lambda)$. We now consider the operator, $\mathcal{J}_n := \Lambda_n^{-1} T_n \Lambda_n \big|_{R(\Lambda)} : R(\Lambda) \to R(\Lambda)$. We notice that since $R(\Lambda_n)$ is invariant under T_n we have that $\sigma_{\mathcal{J}_n} = \{\mu_n^{(1)}, \dots, \mu_n^{(m)}\}$ and the algebraic and geometric multiplicity of each eigenvalue are the same for T_n and $\mathcal{J}_n$. Considering as $\mathcal{J}$ the restriction of T on $R(\Lambda)$ we see that $\sigma(\mathcal{J}) = \{\mu\}$. Notice now that $\operatorname{tr}(\mathcal{J}) = m\mu$ and $\operatorname{tr}(\mathcal{J}_n) = \sum_{j=1}^{m} \mu_n^{(j)}$ and therefore since $\mathcal{J}$ and $\mathcal{J}$ are defined on the same space we have that $\mu - \frac{1}{m}\sum_{j=1}^{m} \mu_n^{(j)}$ is equal to $\frac{1}{m}\operatorname{tr}(\mathcal{J} - \hat{\mathcal{J}})$. Let us now consider a a basis $\phi_1, \dots, \phi_m$ of $R(\Lambda)$ and let $\phi_1^*, \dots, \phi_m^*$ be its dual basis, then we know that

$$\mu - \sum_{j=1}^{m} \mu_n^{(j)} = \frac{1}{m}\operatorname{tr}(\hat{\mathcal{J}} - \hat{\mathcal{J}}_n) = \frac{1}{m}\sum_{j=1}^{m} \langle \phi_j^*, (\mathcal{J} - \mathcal{J}_n)\phi_j \rangle . \qquad (*)$$

Notice that $\phi_j^* \in R(\Lambda_n)^*$ but it can be extended to the all X^*. In fact, since the space X can be decomposed into $X = R(\Lambda) + N(\Lambda)$ we might write any $x \in X$ as $x = r + m$ respectively belonging to $R(\Lambda)$ and $N(\Lambda)$. We assign to $\langle \phi_j^*, x \rangle$ the value $\langle \phi_j^*, r \rangle$, and clearly this extension is bounded. Notice that $\langle (T^* - \mu I)^\alpha \phi_j^*, x \rangle = \langle \phi_j^*, (T - \mu I)^\alpha x \rangle$ and that if $x \in N((T - xI)^\alpha)$ then the duality action must vanish implying that ϕ_j^* are the generalised eigenvectors of T^* corresponding to μ. Now using the fact that $T_n \Lambda_n = \Lambda_n T_n$ and the previously proven fact that $\Lambda_n^{-1}\Lambda_n$ is an identity on $R(\Lambda)$ we see that:

$$|\langle \phi_j^*, (\mathcal{J} - \mathcal{J}_n)\phi_j \rangle| = |\langle \phi_j^*, T\phi_j - \Lambda_n^{-1} T_n \Lambda_n \phi_j \rangle| = \langle \phi_j^*, \Lambda_n^{-1} \Lambda_n (T - T_n)\phi_j \rangle \leq \|\Lambda_n^{-1}\Lambda\| \|\phi_j\| \|\phi_j^*\| \|(T - T_n)|_{R(\Lambda)}\|$$

Combining $(*)$ with this last inequality gives us the desired estimate, simply observing that in the same way we have proven that $\{T_n\}$ was collectively compact we can also prove that $\{\Lambda_n\}$ is such.

PROPOSITION Let $\{\phi_i\}_{i=1}^{m}$ be a basis of $R(\Lambda)$ and let $\{\phi_i^*\}_{i=1}^{m}$ be its dual basis, then there exist a constant $C > 0$, such that the following estimate holds:

$$\left|\mu - \frac{1}{m}\sum_{j=1}^{m} \mu_n^{(j)}\right| \leq \frac{1}{m}\sum_{j=1}^{m} |\langle \phi_j^*, (T - T_n)\phi_j \rangle| + C \, \|(T - T_n)|_{R(\Lambda)}\| \, \|(T^* - T_n^*)|_{R(\Lambda^*)}\|$$

PROOF From the previous theorem we have that,

$$|\langle \phi_j^*, (\mathcal{J} - \hat{\mathcal{J}}) \rangle = |\langle \phi_j^*, T\phi_j - \Lambda_n^{-1} T_n \Lambda_n \phi_j \rangle| = |\langle (\Lambda_n^{-1}\Lambda_n)^* \phi_j^*, (T - T_n)\phi_j \rangle|$$

$$= |\langle \phi_j^*, (T - T_n)\phi_j \rangle + \langle (\Lambda_n^{-1}\Lambda_n)^* \phi_j^* - \phi_j^*, (T - T_n)\phi_j \rangle| . \qquad (*)$$

We now denote $L_n := \Lambda_n^{-1}\Lambda_n$, this is the projection on $R(\Lambda)$ along $N(\Lambda_n)$, thus L_n^* is the projection on $N(\Lambda_n^\perp) = R(\Lambda_n^*)$ along $R(\Lambda)^\perp = N(\Lambda^*)$. Since ϕ_j^* are eigenfunction for the dual problem we know that $L_n^* \phi_j^* - \phi_j^* = (L_n^* - \Lambda^*)\phi_j^*$, thus we can apply one of the previous lemma to obtain

$$\|L_n^* \phi_j^* - \phi_j^*\| \leq \left(\|\Lambda^*\| \|I - \Lambda^*\| \hat{\delta}(R(\Lambda^*), R(\Lambda_n^*))\right)\left(1 - \|I - \Lambda^*\| \hat{\delta}(R(\Lambda^*), R(\Lambda_n^*))\right)^{-1} \|\phi_j^*\|.$$

Notice that we are in the hypothesis of the lemma applying one of the previous THEOREM, to show that $\lim_{n\to\infty} \delta(R(\Delta^*), R(\Delta_n^*)) = 0$. Furthermore once again applying the same theorem we get,

$\| L_n^* \phi_j^* - \phi_j^* \| \le C_3 \| (T^* - T_n^*)|_{R(\Delta^*)} \|$ hence we have that $(*)$ becomes:

$$\Big| \mu - \frac{1}{m} \sum_{j=1}^{m} \mu_n^{(j)} \Big| \le \frac{1}{m} \sum_{j=1}^{m} |\langle \phi_j^*, (T - T_n)\phi_j \rangle| + C_3 \|(T-T_n)|_{R(\Delta)}\| \, \|(T^* - T_n^*)|_{R(\Delta^*)}\| .$$

THEOREM Let α be the ASCENT of μ, consider $\{\phi_i\}_{i=1}^{m}$ a basis for $R(E)$ and let $\{\phi_i^*\}_{i=1}^{m}$ be its dual basis. Then there is a constant C such that

$$|\mu - \mu_n^{(j)}|^{\alpha} \le C \sum_{i,j=1}^{m} |\langle \phi_i^*, (T-T_n)\phi_j \rangle| + \|(T-T_n)|_{R(\Delta)}\| \, \|(T^* - T_n^*)|_{R(\Delta^*)}\|$$

PROOF We begin considering an eigenvectors w_n associated with $\mu_n^{(j)}$. We choose w_n^* belonging to $N((T^* - \mu I)^{\alpha})$ in such a way that $\langle w_n^*, w_n \rangle = 1$ and $\|w_n^*\|$ is bounded. This can be extended to the all X^* as done in one of the previous theorems. Notice that such extension procedure is such that $\|w_n^*\| \le \|\Delta\|$. Since the ascent of μ is α we know $(T-\mu I)^{\alpha} w_n = 0$

$$|\mu - \mu_n^{(j)}|^{\alpha} = |\langle w_n^*, (\mu - \mu_n^{(j)})^{\alpha} w_n \rangle| = |\langle w_n^*, ((T-\mu I)^{\alpha} - (\mu I - \mu_n^{(j)} I)^{\alpha}) w_n|$$

$$\overset{\circledast}{=} \Big|\Big\langle w_n^*, \sum_{j=0}^{\alpha-1} (\mu - \mu_n^{(j)})^{j} (T - \mu I)^{\alpha-1-j} (T - \mu_n^{(j)} I) w_n \Big\rangle\Big|$$

$\circledast$ we have used the identity $a^n - b^n = (a-b)\sum_{j=0}^{n-1} a^{n-j-1} b^j$.

$$\le \sum_{j=0}^{\alpha-1} |\mu - \mu_n^{(j)}|^{j} \, |\langle (T^* - \mu)^{\alpha-1-j} w_n^*, (T - \mu_n^{(j)} I) w_n \rangle|$$

$$\le \sum_{j=0}^{\alpha-1} |\mu - \mu_n^{(j)}|^{j} \max_{\substack{\phi^* \in R(\Delta^*) \\ \|\phi^*\|=1}} |\langle \phi^*, (T - \mu_n^{(j)} I) w_n \rangle| \, \|T^* - \mu I\|^{\alpha-1-j} \, \|w_n^*\| \qquad (**)$$

Now we notice that for any $\phi^* \in R(\Delta^*)$ with $\|\phi^*\| = 1$ we have:

$$|\langle \phi^*, (T - \mu_n^{(j)} I) w_n \rangle| = |\langle \phi^*, (\hat{\jmath} - T) w_n \rangle| = |\langle \phi^*, \Delta_n^{-1} \Delta_n (T_n - T) w_n \rangle| \le |\langle \phi_j^*, (T_n - T) w_n \rangle| + |\langle L_n^* \phi^* - \phi_j^*, (T_n - T) w_n \rangle|$$

$$\le |\langle \phi_j^*, (T_n - T) w_n \rangle| + C \|(T - T_n)|_{R(\Delta)}\| \, \|(T^* - T_n^*)|_{R(\Delta^*)}\| ,$$

and thus it exist a constant C such that for any $\phi^* \in R(\Delta^*)$ with $\|\phi^*\| = 1$:

$|\langle \phi^*, (T - \mu_n^{(j)} I) w_n \rangle| \le C' \sum_{i,j=1}^{m} |\langle \phi_j^*, (T_n - T)\phi_j \rangle$ and substituting this estimates inside $(**)$ we have concluded, in fact we have obtained:

$$|\mu - \mu_n^{(j)}|^{\alpha} \le \sum_{j=0}^{\alpha-1} |\mu - \mu_n^{(j)}|^{j} \Big[C' \sum_{i,j=1}^{m} |\langle \phi_j^*, (T_n - T)\phi_j \rangle| + C \|(T - T_n)|_{R(\Lambda)}\| \, \|(T^* - T_n^*)|_{R(\Lambda^*)}\| \Big] .$$

picking n sufficiently large that $|\mu - \mu_n^{(j)}| < 1$ allows us to conclude.

THEOREM Let μ_n be an eigenvalue of T_n such that $\lim_{n\to\infty} \mu_n = \mu$. Suppose that for each n that w_n is a unit vector belonging to $N((T_n - \mu_n)^k)$ for some $k \in \mathbb{N}$ and $0 < k \le \alpha$. Then for any $\ell \in \mathbb{N}$ such that $k \le \ell \le \alpha$ there exist a vector $x_n \in R(\Lambda)$ such that $\|x_n - w_n\| \le C \|(T - T_n)|_{R(\Lambda)}\|^{(\ell-k+1)\alpha^{-1}}$.

PROOF We begin observing that since $N((T-\mu I)^{\ell})$ is finite dimensional then there exist a closed subspace M of X such that $X = N((T-\mu I)^{\ell}) \oplus M$ and $\forall y \in R((T-\mu I)^{\ell})$ the equation $(T-\mu I)^{\ell} x = y$ is uniquely solvable in M thus $(T-\mu I)^{\ell}|_M : M \to R((T-\mu I)^{\ell})$ is bijective hence $(T-\mu I)^{-\ell}$ exists and $(T-\mu I)^{-\ell} \in B(R((T-\mu I)^{\ell}), M)$, thus $\exists C$ such that $\|x\| \le C \|(T-\mu I)^{\ell} x\|$ $\forall x \in M$.

Let $x_n = \Pi w_n$, i.e. the projection on $N((T-\mu I)^{\ell})$ along M and observe that $(T-\mu I)^{\ell} x_n = 0$ and $w_n - x_n \in M$ hence $\|w_n - x_n\| \le C \|(T-\mu I)^{\ell}(w_n - x_n)\|$.

By the first among the previous theorems we know that $\exists \tilde{x}_n \in R(\Lambda)$ such that the control $\|w_n - \tilde{x}_n\| \le C \|(T-T_n)|_{R(\Lambda)}\|$. Thus we notice that the following inequalities must hold,

$$\| ((T-\mu I)^{\ell} - (T_n-\mu I)^{\ell}) w_n \| \underset{*}{=} \| \sum_{j=0}^{\ell-1} (T_n-\mu I)^j (T_n-T)(T-\mu I)^{\ell-j-1} (w_n - \tilde{x}_n + \tilde{x}_n) \| \leq C \| (T_n-T)|_{R(\Delta)} \|.$$

$$\| (T_n-\mu I)^{\ell} w_n \| = \| \sum_{j=0}^{\ell} \binom{\ell}{j} (\mu-\mu_n)(T_n^{\ell-j}-\mu I)^{\ell-j} w_n \| = \| \sum_{j=\ell-k+1}^{\ell} \binom{\ell}{j} (\mu-\mu_n)(T_n^{\ell-j}-\mu I)^{\ell-j} w_n \|$$

$$\leq C |\mu-\mu_n|^{\ell-k+1}.$$

Hence we have,

$$\|w_n - x_n\| \leq C \| (T-\mu I)^{\ell} w_n \| = C \| ((T-\mu I)^{\ell} - (T_n-\mu I)^{\ell}) w_n + (T-\mu)^{\ell} w_n \| \leq C (\| (T-T_n)|_{R(\Delta)} + |\mu-\mu_n|^{\ell-k+1}).$$

applying the previous theorem we conclude.

* MISSING APPLICATION TO SELF-ADJOINT CASE, i.e. BABUSKA-OSBORN.