Galerkin Discretisation of Eigenvalue Problems

We have spent a great deal of time discussing finite rank approximation of a compact operator, yet most PDE are not bounded ENDOMORPHISM, thus one might think that presented so far is not applicable to PDEs. Yet once again the "inverse trick" we presented in the previous section comes to our aid. Given two bilinear forms, i.e. $k: X \times X \to \mathbb{R}$ and $m^*: X \times X \to \mathbb{R}$ we consider the variational eigenvalue problem: find $(u,\lambda) \in X \times \mathbb{C}$ such that $k(u,v) = \lambda\, m(u,v)$ for any $v \in X$. We will from now on assume that the "source problem" associated with the above eigenvalue problem is well posed, i.e. $\forall f \in X$ it $\exists!\, u \in X$ such that $k(u,v) = m(f,v)$ for any $v \in X$. Under this hypothesis we can introduce the solution operator, i.e. $T: X \to X$ $\quad f \mapsto u$ such that $k(Tu, v) = (f, v)$ $\forall v \in X$. Furthermore we will work under the hypothesis that $T \in K(X,X)$. Notice that this hypothesis is often verified by many PDE for which sufficient regularity of the solution can be proven. For example any ELLIPTIC PROBLEM with sufficiently regular coefficient will satisfy this condition [GRISVARD], in particular in a previous example we have shown this was the case for the Laplace equation in weak-form.

Remark For the operator T to be compact in X, it is sufficient that the bilinear form $k: X \times X \to \mathbb{R}$ is continuous with respect to the norm of X and that it exist a norm $|||\cdot|||$, such that any we can extract from any bounded sequence with respect to $\|\cdot\|_X$ a Cauchy subsequence with respect to $|||\cdot|||$, for which the following bound hold: $|b(u,v)| \leq |||u|||\, \|v\|$ $\forall u,v \in X$. We remark that by no-means this is a necessary condition for the solution operator T to be compact and it might be convenient to work with a compact T even without the fact that the previous proposition applies. A typical example is when working with the $H(\nabla \times, \Omega)$ space.

Now that we are armed with a compact solution operator we want to relate the eigenvalues of the compact operator μ to the one of the variational eigenvalue problem. To this end we observe that if u is an eigen function of the variational problem associated with an eigen function u, we have $a(u,v) = \lambda\, m(u,v) = \lambda\, a(Tu,v)$ which implies that $a(u - \lambda Tu, v) = 0$ for any $v \in V$ thus by the solvability condition we get $\lambda Tu = u$ hence $Tu = \frac{1}{\lambda} u$ implying that $1/\lambda$ is an eigen value of T and the eigen functions of T are also the eigen functions of the variational eigenvalue problem. We now consider a sequence of Galerkin discretisation, i.e. we consider a sequence of finite dimensional subspaces $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \subseteq X$ and $\dim(X_n) = n$. Given a $x \in X$ we will denote $T_n: X \to X_n$ the sequence of solution operators associated with the Galerkin discretisation, i.e. $\quad k(T_n f, v_n) = m(f, v_n)$ $\forall v_n \in X_n$. Notice that since $T_n: X \to X_n \subseteq X$ is an ENDOMORPHISM.

We will further assume that the Galerkin method is convergent, i.e. $\forall x \in X$ we assume that $\lim_{n\to\infty} \|T_n x - Tx\| = 0$.

Remark Notice that when k and m are symmetric bilinear forms associated inducing a scalar product on X we can simply rewrite $T_n = \Pi_n T$, where Π_n is the orthogonal projection with respect to the inner product induced by $k: X \times X \to \mathbb{R}$. This is because from the discrete and continuous variational formulations we have, $k(T_n f, v_n) = m(f, v_n) = k(T_n f, v_n)$. Using the Hilbert projection theorem we deduce that it $\exists!$ solution to the discrete variational problem. Furthermore, $\|Tf - T_n f\| \leq \inf_{v_n \in X} \|u - v_n\|$ hence from the approximation property of X_n one can conclude that the Galerkin discretisation is convergent. In the non self-adjoint case the well-posedness of the Galerkin discretisation and its convergence have to be taken as hypothesis and proven on a case by case base.

Notice that the key struct exploited in the previous remark is that we can express T_n as an orthogonal projection of T with respect to the inner product induced by $k: X \times X \to \mathbb{R}$, i.e. $T_n = \Pi_n T$.

Theorem (Kolata) Consider a sequence $\{X_n\}_{n\in\mathbb{N}}$ of discrete subspaces of X, and $T \in K(X,X)$. Let Π_n be projection on X_n by the inner product Π_n, and assume that $\forall u \in X$ $\lim_{n\to\infty} \frac{\|u - \Pi_n u\|}{\|u\|_V} = 0$ then $\lim_{n\to\infty} \|T - T_n\| = 0$, where $T_n = \Pi_n T$, and V is a subspace of X such that $T \in K(X,V)$.

Proof First we prove that the sequence $\{(I - \Pi_n)\}_{n\in\mathbb{N}}$ is bounded. We define $C_n(u) := \|(I-\Pi_n)u\|_X \|u\|_V^{-1}$. From our hypothesis we know that $C_n(u) \searrow 0$ and thus it $\exists n^*$ such that $K(u) = \sup_{n > n^*} c_n(u) < \infty$. Hence by the Banach-Steinhaus theorem the operator $(I-\Pi_n) \in B(V,X)$. We now consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $\|f_n\|_X = 1$ and $\|T-T_n\| = \|(T-T_n)f_n\|$ which exists because we just proved that $T_n \in \mathcal{B}(X,X)$. Notice now that $\{f_n\}$ is bounded in H and T is compact from H to V, thus we can extract a converging subsequence in the range of T. We denote by a slight abuse of notation the subsequence $\{f_n\}$ and its limit w.

Notice now that $T_n v = \Pi_n T v = \Pi_n w$, thus: $\quad \Pi_n : X \to X_n$

$$\|(I-\Pi_n)Tf_n\|_H \leq \|(I-\Pi_n)(Tf_n - w)\|_X + \|(I-\Pi_n)w\|_X = C\underbrace{\|Tf_n - w\|_X}_{\searrow 0 \text{ because we extracted converging subsequence.}} + \overbrace{\|(I-\Pi_n)w\|_X}^{\text{Vanish by hypothesis.}} \searrow 0$$

The Kolata argument can be generalised to the setting of NON-NORMAL setting provided we can rewrite T_n as $\Pi_n T$, and $\forall u \in X$ $\lim_{n\to\infty} \|u - \Pi_n u\| \|u\|_V^{-1} = 0$, for a subspace V such that $T \in K(X,V)$ is compact. Thinking of the inverse of T_n this implies $T_n^{-1} = (\Pi_n T)^{-1} = T^{-1}\Pi_n^{-1}$ hence we are asking that it $\exists$ a projection from the discrete space to the continuous one such that it's inverse has good approximation properties.

REMARK Notice once more that we only ask that $T \in K(X,V)$ and not that $V \subset\subset X$, which comes particularly handy when working with H-curl.

The Kolata argument is an extremely powerful trick because it allows to transform pointwise convergence in uniform convergence and thus use the result previously proven. Things becomes more subtle if the variational eigenvalue problem has a saddle point structure.