Consider the one-dimensional advection-diffusion eigenvalue problem, i.e.

$$\eta - u''(x) - au'(x) = \lambda u(x), \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0, \tag{1}$$

where λ is the eigenvalue and $u(x)$ is the eigenfunction. The weak form of the problem is given by:

$$\eta \int_0^\pi u'(x)v'(x)\,dx - a\int_0^\pi u(x)v'(x)\,dx = \lambda \int_0^\pi u(x)v(x)\,dx \tag{2}$$

for all $v \in H_0^1(0, \pi)$, where $H_0^1(0, \pi)$ is the Sobolev space of functions that are square integrable and have square integrable weak derivatives, with boundary conditions $u(0) = u(\pi) = 0$.

1. Compute the first ten eigenvalues and eigenfunctions of the problem, with $\eta = 1$, using the linear finite element method and the Firedrake library. [Hint: *the exact eigenvalues are given by* $\lambda_n = +\frac{1}{4}\eta^{-1} + \eta n^2$ *and the eigenfunctions are given by* $u_n(x) = e^{-\frac{1}{2}\eta^{-1}x}\sin(n\pi x)$, *where* $n = 1, 2, \ldots$. To see why this is the case look at Lecture 4.]
2. What is the order of convergence of the eigenvalues and eigenfunctions? Check that it is the same as the theoretical order of convergence. [Hint: *The theoretical order of convergence can be computed using Osborn theory, see. Lecture 3.*]
3. Compute different the first eigenfunction for different values of η, e.g. $\eta \in \{1, 0.5, 0.1\}$. How do the eigenvalue and the eigenfunctions change as η changes? Does the order of convergence of the eigenvalues and eigenfunctions change as η changes? [Hint: *The advection-diffusion problem is a singular perturbation problem as η goes to zero. Thus, we expect the formation of boundary layers.*]

Consider the one-dimensional Helmholtz equation associated with the advection-diffusion operator, i.e.

$$-\eta u''(x) - au'(x) - \omega^2 u(x) = f, \quad x \in (0, 1), \quad u(0) = u(\pi) = 0, \tag{3}$$

where ω is the wavenumber. The weak form of the problem is given by:

$$\eta \int_0^\pi u'(x)v'(x)\,dx - a\int_0^\pi u(x)v'(x)\,dx - \omega^2 \int_0^\pi u(x)v(x)\,dx = \int_0^\pi f v(x)\,dx \tag{4}$$

for all $v \in H_0^1(0, \pi)$, where $H_0^1(0, \pi)$.

4 Let $\eta \in \{1, 0.5, 0.1\}$ and pick as functions $f(x) = \sin(\pi x)$ and $f_\delta(x) = \sin(\pi x + \delta)$. Where δ is a small perturbation, e.g. $\delta = 0.1$. Compute the solution of the problem for $\omega = 2 + 2i$ for the two data f and f_δ. How do the two solutions compare as η changes?

Consider the two-dimensional advection-diffusion eigenvalue problem, i.e.

$$-\eta\Delta u(x) - \mathbf{w}\cdot\nabla u(x) = \lambda u(x), \quad x \in (0,1)^2, \quad u(0) = u(1) = 0, \tag{5}$$

where λ is the eigenvalue and $u(x)$ is the eigenfunction and $\mathbf{w} = (w_1, w_2)$ is the advection velocity.

5 Compute the first five eigenfunctions of the problem when $\eta = 1$ and $\mathbf{w} = (1,0)$ using the linear finite element method and the Firedrake library.

6 Discuss the impact of varying $\eta \in \{1, 0.5, 0.1\}$ on the eigenfunctions and eigenvalues.

7 Analyze the convergence behavior of the eigenvalues as η changes, e.g. $\eta \in \{1, 0.5, 0.25\}$.