# On the convergence of eigenvalues for mixed formulations


PAPER BY:
*Daniele Boffi*, Franco Brezzi* and Lucia Gastaldi**

PRESENTED BY:
*Umberto Zerbinati*

**University of Pavia*


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# Virtual Handouts



All code to reproduce the examples shown in this slides can be found in the following git repository:

`https://github.com/UZerbinati/BrezziBoffiGastaldi`

- Solved Bernstein's minimal surface problem together with Enrico Bombieri.
- Solved 19th Hilbert problem, regarding the regularity of elliptic PDE.
- He is the father of modern calculus of variation and he is responsible for introducing the notion of Γ-convergence.

**Enio De Giorgi**
1928–1996

- Two Hilbert spaces $\Phi$ and $\Xi$ are considered.
- Two **continuous** bilinear forms are also considered,

$$a(\cdot,\cdot) : \Phi \times \Phi \to \mathbb{R}, \qquad b(\cdot,\cdot) : \Phi \times \Xi \to \mathbb{R}.$$

$$A : \Phi \to \Phi^*, \; (A\phi) : \Phi \to \mathbb{R}, \; (A\phi)\varphi \mapsto a(\phi,\varphi),$$
$$B : \Phi \to \Xi^*, \; (B\phi) : \Xi \to \mathbb{R}, \; (B\phi)\xi \mapsto b(\phi,\xi),$$

- We assume $a(\cdot,\cdot)$ is **symmetric** and **positive semidefinite**.

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*Given any* $(f, g) \in \Phi^* \times \Xi^*$ *find* $(\psi, \chi) \in \Phi \times \Xi$ *such that*

$$\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = \langle f, \varphi \rangle & \forall \varphi \in \Phi \\ b(\psi, \xi) = \langle g, \xi \rangle & \forall \xi \in \Xi \end{cases} \tag{1}$$

*Given any* $(f, g) \in \Phi^* \times \Xi^*$ *find* $(\psi, \chi) \in \Phi \times \Xi$ *such that*

$$\begin{cases} A\psi + B^*\chi = f \\ B\psi = g \end{cases}$$

## Banach Closed Range **Theorem**

Given a closed linear operator $B : \Phi \to \Xi^*$, the following statements are equivalent:

- $Range(B)$ is closed in $\Xi^*$,
- $Range(B) = \left[Ker(B^T)\right]^0$, where $Z^0$ is the the **polar** set of $Z$, i.e. $Z^0 = \left\{ f \in Z^* \; s.t \; \langle f, z \rangle = 0 \; \forall z \in Z \right\}$.
- it exists $L_B \in \mathcal{L}\Big( Range(B), Ker(B)^\perp \Big)$ and $\beta \geq 0$ such that:

$$\beta \| L_B g \|_\Phi \leq \| g \|_{\Xi^*}, \; \forall g \in Range(B).$$

## Brezzi's **Theorem**

Assuming that the range of $B$ is $\Xi^*$ and that $a(\cdot,\cdot)$ is coercive in the kernel of $B$, then it exists one and only one solution to (1).

**How do can one verify that $B$ is a onto ?**

## Inf-Sup **Condition**

The operator $B$ is surjective if and only if it exists $\beta > 0$ such that

$$\inf_{\xi\in\Xi}\sup_{\varphi\in\Phi}\frac{b(\varphi,\xi)}{\|\varphi\|_\Phi\|\xi\|_\Xi} \geq \beta$$

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- We introduce two discrete spaces $\Phi_h \subset \Phi$ and $\Xi_h \subset \Xi$.
- Previous result holds also for the discrete problem,

*Given any* $(f, g) \in \Phi^* \times \Xi^*$ *find* $(\psi_h, \chi_h) \in \Phi_h \times \Xi_h$ *such that*

$$\begin{cases} a(\psi_h, \varphi_h) + b(\varphi_h, \chi_h) = \langle f, \varphi_h \rangle & \forall \varphi \in \Phi_h \\ b(\psi_h, \xi_h) = \langle g, \xi_h \rangle & \forall \xi \in \Xi_h \end{cases} \tag{2}$$

- The pair $(\Phi_h, \Xi_h)$ is **inf-sup stable** if it exists $\beta_h$ independent from $h$ such that:

$$\inf_{\xi_h \in \Xi_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \xi_h)}{\|\varphi_h\|_\Phi \|\xi_h\|_\Xi} \geq \beta_h$$

We introduce the **solution operator**, i.e.

$$S : \Phi^* \times \Xi^* \to \Phi \times \Xi \ s.t\ S(f,g) = (\psi, \chi) \ as\ in\ (1),$$
$$S_h : \Phi_h^* \times \Xi_h^* \to \Phi_h \times \Xi \ s.t\ S_h(f,g) = (\psi_h, \chi_h) \ as\ in\ (2).$$

**Proposition** Necessity of the Inf-sup

If it exists a constant $C > 0$ such that for all $(f,g) \in \Phi^* \times \Xi^*$ and for all $h > 0$

$$\|S_h(f,g)\|_{\Phi\times\Xi} \leq C\Big(\|f\|_{\Phi_h^*} + \|g\|_{\Xi_h^*}\Big) \tag{3}$$

then the bilinear form $a(\cdot,\cdot)$ is elliptic in the kernel of $B$ and the pair $(\Phi_h, \Xi_h)$ is **inf-sup** stable.

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When dealing with an mixed problem such that $a(\cdot, \cdot)$ is elliptic in the kernel the **inf-sup stability** condition is not only **sufficient** it is also **necessary**.

### Remark

When proving the existence and uniqueness of solution for 1 and 2, hypothesis can be weaken, i.e. we can require $A : Ker(B) \to Ker(B)^*$ to be an isomorphism.

We are now ready to introduce an eigenvalue problem. Given an Hilbert space $H$ and a selfadjoint compact operator $T : H \to H$, we call eigenvalue of $T$ the $\lambda \in \mathbb{R}$ such that

$$\lambda Tu = u \text{ with } u \in H \setminus \{0\}.$$

In particular it is well known that for the above described operator $T$ it exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ such that

$$\lambda_i T u_i = u_i,$$
$$\lim_{i\to\infty} \lambda_i = +\infty \text{ and } \lambda_i \geq 0 \; \forall i \in \mathbb{N}. \tag{4}$$

We now consider for all $h > 0$ the selfadjoint non negative operator $T_h : H \to H$, with finite range $H_h$. Let's denote $N(h)$ is the dimension of $H_h$. We are interested in the eigenvalues,

$$\lambda^h T u^h = u^h \textit{ with } u^h \in H_h \backslash \{0\}.$$

The same characterization of the eigenvalue presented above holds also in the discrete case.

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If we assume that the discrete approximation operator $T_h$ converges to $T$ with respect to the norm of $\mathcal{L}(H,H)$, i.e.

$$\lim_{h\to 0} \| T - T_h \|_{\mathcal{L}(H,H)} = 0$$

then $\forall \varepsilon > 0$ and $\forall n \in \mathbb{N}$ it exists $h_0 > 0$ such that $\forall h > h_0$

$$\max_{i=1,\dots,m(N)} \left| \lambda_i - \lambda_i^h \right| \leq \varepsilon, \tag{5}$$

$$\delta\left( \oplus_{i=1}^{m(N)} E_i, \oplus_{i=1}^{m(N)} E_i^h \right) \leq \varepsilon. \tag{6}$$

$m(N)$ is the number of eigenvalues corresponding to $N$ distinct ones, $E_i = \langle u_i \rangle$ and $E_i^h = \langle u_i^h \rangle$. The converse also holds true.

We consider two Hilbert space $H_\Phi$ and $H_\Xi$ such that we cam identify $H_\Phi$ with $H_\Phi^*$, $H_\Xi$ with $H_\Xi^*$ and

$$\Phi \subset H_\Phi \subset \Phi^*, \qquad \Xi \subset H_\Xi \subset \Xi^*.$$

**Proposition** Convergence of Discrete Eigenvalue Problem

Assuming that $a(\cdot,\cdot)$ is elliptic in the kernel of $B_h$ and the discrete inf-sup condition holds then $S_h$ converges in $\mathcal{L}(H_\Phi, H_\Xi)$ to $S$ if and only if $S : H_\Phi \times H_\Xi \to H_\Phi \times H_\Xi$ is compact. The converse holds true.

**Compactness plays a key role in eigenvalue problem, for more detail check Boffi, Acta Numerica 2010.**

## Motivating Example – Stokes Eigenvalue Problem with Q1-P0

```
msh = UnitSquareMesh(10,10,quadrilateral=True)
V = VectorFunctionSpace(msh, "Q", 1)
Q = FunctionSpace(msh, "DG", 0)
X = V*Q
u,p = TrialFunctions(X)
v,q = TestFunctions(X)
a = (inner(grad(u), grad(v)) - inner(p, div(v))
  + inner(div(u), q))*dx+1e8*inner(u,v)*ds
m = inner(u,v)*dx
sol = Function(X)
```

# The Story Doesn't End Here

## Motivating Example – Stokes Eigenvalue Problem Q1-P0

```
A = assemble (a)
M = assemble (m)
Asc, Msc = A.M.handle, M.M.handle
E = SLEPc.EPS().create()
E.setType(SLEPc.EPS.Type.ARNOLDI)
E.setProblemType(SLEPc.EPS.ProblemType.GHEP);
E.setOperators(Asc,Msc)
PC = ST.getKSP().getPC();
PC.setType("svd");
E.setST(ST);
E.solve();
```

## The Story Doesn't End Here
### Motivating Example – Stokes Eigenvalue Problem Q1-P0

| N | Reference | Q1-P0 |
|---|---|---|
| 1 | 52.34468 | 53.56885 |
| 2 | 92.12438 | 97.57386 |
| 3 | 92.12438 | 97.57386 |
| 4 | 128.209 | 97.573867 |

- The numerical experiment for the Q1-P0 are obtained using a $10 \times 10$ uniform square grid, while the reference value are obtained using Hood-Taylor finite element pair on a $20 \times 20$ square mesh.
- As $h \to 0$ we would see a degraded rate of convergence.

# Two Type Of Problems
## The $(f \quad 0)$ Example



More often the note in practice when we are interested in eigenvalue problem where either $f$ or $g$ is zero. For example we call problem of type $(f \quad 0)$,

$$\textit{Find } (\psi,\chi) \in \Phi \times \Xi \textit{ and } \lambda \in \mathbb{R} \textit{ such that}$$

$$\begin{cases} a(\psi,\varphi) + b(\varphi,\chi) = \lambda\langle\psi,\varphi\rangle & \forall\varphi \in \Phi, \\ b(\psi,\xi) = 0 & \forall\xi \in \Xi. \end{cases} \tag{7}$$

Which <u>**can not**</u> be cast as an eigenvalue problem of the form of (1).

# Two Type Of Problems

## Problem of Type $\begin{pmatrix} f & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & g \end{pmatrix}$ Type.

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To recast (7) as an eigenvalue problem we need to introduce

$$\begin{aligned} C_\Phi : \Phi^* &\to \Phi^* \times \Xi^* \\ f &\mapsto (f, 0) \end{aligned} \qquad\qquad \begin{aligned} C_\Phi^* : \Phi \times \Xi &\to \Phi \\ (\varphi, \xi) &\mapsto \varphi \end{aligned}$$

then we can study the eigenvalue problem corresponding to

$$T_\Phi := C_\Phi^* \circ S \circ C_\phi : \Phi^* \to \Phi \tag{8}$$

- What are the necessary and sufficient conditions to solve an eigenvalue problem like (7) ?

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**Proposition** Existence of Solutions

If $a(\cdot,\cdot)$ is elliptic in the kernel of $B_h$, then problem (7) admits at least one solution $(\psi_h, \chi_h)$. Moreover $\psi_h$ in uniquely determined by $f$ and

$$\|\psi_h\|_\Phi \leq C\|f\|_{\Phi_h^*}.$$

Furthermore if it exists $C > 0$ such that for every $h > 0$ and for every $(\psi_h, \chi_h, f) \in \Phi_h \times \Xi_h \times \Phi^*$ the above inequality is verified then the operator $T_\Phi^h$ is defined for all element in $\Phi$ and $a(\cdot,\cdot)$ elliptic in the kernel of $B_h$.

## Definition Weak Approximability

Let $\Xi_0^H$ be the range of $C_\Xi^* \circ S \circ C_\Phi$. We say that $\Xi_0^H$ verifies the **weak approximability** if for every $\chi \in \Xi_0^H$

$$\sup_{\varphi_h \in Ker(B_h)} \frac{b(\varphi_h, \chi)}{\|\varphi\|_\Phi} \leq \omega_1(h)\|\chi\|_{\Xi_0^H}, \ \lim_{h \to 0} \omega_1(h) = 0.$$

## Remark

The above definition is an approximability condition in fact using the fact that $b(\varphi_h, \chi^I) = 0$ for all $\chi^I \in \Xi_h$ to rewrite the weak approximability as: for all $\chi \in \Xi_0^H$ $\inf_{\chi^I \in \Xi_h} \|\chi - \chi^I\|_\Xi \leq \omega_1(h)\|\chi\|_{\Xi_0^H}$.

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**Definition** Strong Approximability

Let $\Phi_0^H$ be the range of $C_\Phi^* \circ S \circ C_\Phi$. We say that $\Phi_0^H$ verifies the **strong approximability** if for every $\psi \in \Phi_0^H$

$$\inf_{\psi^I \in Ker(B_h)} \left\| \psi - \psi^I \right\|_\Phi \leq \omega_2(h) \|\psi\|_{\Phi_0^H}, \ \lim_{h \to 0} \omega_2(h) = 0.$$

# Problem of Type $(f \quad 0)$

## Necessary and Sufficient Conditions for Convergence

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**Proposition** Convergence

If $a(\cdot,\cdot)$ is elliptic in the kernel of $B_h$ and the weak approximability of $\Xi_0^H$ and strong approximability of $\Phi_0^H$ are verified, then for all $f \in H_\Phi$

$$\left\| T_\Phi f - T_\Phi^h f \right\|_\Phi \leq \omega_3(h), \ \lim_{h \to 0} \omega_3(h) = 0. \tag{9}$$

Vice versa if the sequence $T_\Phi^h$ is bounded in $\mathcal{L}(\Phi^*, \Phi)$ and converges uniformly to $T_\Phi$ in $\mathcal{L}(\Phi^*, \Phi)$ then $a(\cdot,\cdot)$ is elliptic in the kernel of $B_h$, moreover the strong and weak approximability conditions are verified respectively for $\Phi_0^H$ and $\Xi_0^H$.

# An additional Example

## A Connection with Charlie's presentation

We solve the Stokes eigenvalue problem using Scott-Vogelious(**ish**) finite element pair and criss-cross. mesh.

```
msh = UnitSquareMesh(5,5,diagonal="crossed")
V = VectorFunctionSpace(msh, "CG", 4)
Q = FunctionSpace(msh, "DG", 3)
X = V*Q
u,p = TrialFunctions(X)
v,q = TestFunctions(X)
a = (inner(grad(u), grad(v)) - inner(p, div(v))
  + inner(div(u), q))*dx+1e8*inner(u,v)*ds
m = inner(u,v)*dx
sol = Function(X)
```

# An additional Example

## A Connection with Charlie's presentation

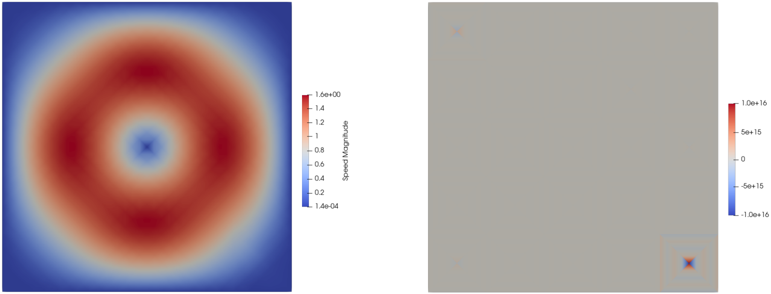


Figure: On the LHS the first mode of the Stokes eigenvalue problem, on the RHS the corresponding pressure computed with a SV(ish) method.

```
msh = SquareMesh(64,64,np.pi,quadrilateral=True)

S = VectorFunctionSpace(msh, "Q", 1)
V = FunctionSpace(msh, "DG", 0)
X = S*V
s,u = TrialFunctions(X); t,v = TestFunctions(X)

a = (inner(s,t)+inner(div(t),u)+inner(div(s),v))*dx
m = -inner(u,v)*dx

bc = DirichletBC (X.sub(0), as_vector([0.0,0.0]) ,
     [1,2,3,4])
A = assemble (a,bcs=bc)
M = assemble (m)
```

| N | Ref | Q1-P0 |
|---|---|---|
| 1 | 1 | 0.99111 |
| 2 | 1 | 0.99111 |
| 3 | 2 | 1.96374 |
| 4 | 4 | 3.96531 |
| 5 | 4 | 3.96669 |
| 6 | 5 | 4.91019 |
| 7 | 5 | 4.91019 |
| 8 | 8 | 7.85488 |
| 9 | 9 | 8.73856 |
| 10 | 9 | 8.73856 |
| 11 | 10 | 8.92934 |

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What happens when $f$ is null rather then $g$ ?
We call problem of this type $\begin{pmatrix} 0 & g \end{pmatrix}$,

$$\textit{Find } (\psi, \chi) \in \Phi \times \Xi \textit{ and } \lambda \in \mathbb{R} \textit{ such that}$$
$$\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = 0 & \forall \varphi \in \Phi, \\ b(\psi, \xi) = -\lambda \langle \chi, \xi \rangle & \forall \xi \in \Xi. \end{cases} \tag{10}$$

Which once again **can not** be cast as an eigenvalue problem of the form of (1).

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To recast (10) as an eigenvalue problem we need to introduce

$$\begin{aligned} C_{\Xi} : \Xi^* &\to \Phi^* \times \Xi^* \\ g &\mapsto (0, g) \end{aligned} \qquad \begin{aligned} C_{\Xi}^* : \Phi \times \Xi &\to \Xi \\ (\varphi, \xi) &\mapsto \xi \end{aligned}$$

then we can study the eigenvalue problem corresponding to

$$T_{\Xi} := C_{\Xi}^* \circ S \circ C_{\Xi} : \Xi^* \to \Xi \tag{11}$$

- What are the necessary and sufficient conditions to solve an eigenvalue problem like (10) ?

**Proposition** Existence of Solutions

If the discrete inf-sup stability condition holds, then problem (10) admits at least one solution $(\psi_h, \chi_h)$. Moreover $\chi_h$ in uniquely determined by $g$ and

$$\|\chi_h\|_{\Xi} \leq C\|g\|_{\Xi_h^*}.$$

Furthermore if it exists $C > 0$ such that for every $h > 0$ and for every $(\psi_h, \chi_h, g) \in \Phi_h \times \Xi_h \times \Xi^*$ the above inequality is verified then the operator $T_{\Xi}^h$ is defined for all element in $\Xi$ and the **discrete** inf-sup condition is verified.

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**Definition** Weak Approximability

Let $\Xi_0^H$ be the range of $C_{\Xi}^* \circ S \circ C_\Phi$. We say that $\Xi_0^H$ verifies the **weak approximability** if for every $(\chi, \varphi_h) \in \Xi_0^H \times Ker(B_h)$

$$b(\varphi_h, \chi) \leq \omega_4(h) \|\chi\|_{\Xi_0^H} \sqrt{a(\varphi_h, \varphi_h)}, \; \lim_{h \to 0} \omega_4(h) = 0.$$

- Notice that this is an approximability condition similar to the one presented for the $\begin{pmatrix} f & 0 \end{pmatrix}$ problems.

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**Definition** Strong Approximability

Let $\Xi_0^H$ be the range of $C_\Xi^* \circ S \circ C_\Xi$. We say that $\Xi_0^H$ verifies the **strong approximability** if for every $\chi \in \Xi_0^H$ it exists $\chi^I \in \Xi_h$ such that,

$$\left\|\chi - \chi^I\right\|_\Xi \leq \omega_5(h)\|\chi\|_{\Xi_H^0}, \lim_{h\to 0} \omega_5(h) = 0. \qquad (12)$$

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### **Definition** Fortin Operator

Given a subspace $\Phi_\Pi$ of $\Phi$ we say an operator $\Pi_h : \Phi_\Pi \to \Phi_h$ is a **Fortin** operator with respect to the bilinear form $b(\cdot, \cdot)$ and the subspace $\Xi_h$ if for all $\varphi \in \Phi_\Pi$ we have that:

$$b(\varphi - \Pi_h \varphi, \xi_h) = 0, \ \forall \xi_h \in \Xi_h.$$

**Proposition** Sufficient Conditions for Convergence

Assuming that it exists a **bounded Fortin operator** $\Pi_h : Range(C_\Phi^* \circ S \circ C_\Xi) \to \Phi_h$ such that for every $\phi \in \Phi_H^0$,

$$\sqrt{a(\varphi - \Pi_h\varphi, \varphi - \Pi_h\varphi)} \leq \omega_6(h)\|\varphi\|_{\Phi_H^0}, \ \lim_{h\to 0} \omega_6(h) = 0. \quad (13)$$

If the weak and strong approximability condition of $\Xi_H^0$ are verified then

$$\left\| T_\Xi f - T_\Xi^h f \right\|_\Xi \leq \omega_7(h)\|g\|_{H_\Xi}, \ \lim_{h\to 0} \omega_7(h) = 0,$$

for any $g \in H_\Xi$.

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**Proposition** Necessary Conditions for Convergence

If the sequence of operators $T_{\Xi}^{h}$ is bounded in $\mathcal{L}(\Xi^*, \Xi)$, it converges to $T_{\Xi}$ in $\mathcal{L}(H_{\Xi}, \Xi)$ and the following bounds holds when $f = 0$,

$$\|\varphi_h\|_{\Phi} \leq C\|g\|_{\Xi},$$

then it exists a **bounded Fortin operator** verifying (13), moreover we have that the discrete inf-sup condition is verified together with the weak and strong approximation property of $\Xi_H^0$.

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- The inf-sup condition is neither necessary nor sufficient when dealing with eigenvalue problem.
- For $(f - 0)$ problem the inf-sup condition is **not necessary**.
- For $(0 - g)$ problem the inf-sup condition is **not sufficient**.
- Why does everything work when dealing with a complex ? **Boffi, Acta Numerica (2010)**.
- A more modern approach which also deals with Kolata argument can be found in **Boffi, Acta Numerica (2010)**.

## Is the Approximation Of Mixed Eigenvalue Problem Closed ?

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- Can we create new element pairs specifically to solve the solve (f – 0) problems ?
- What happens if we use Babuska version of the inf-sup conditions ?

Boffi, D. (2010).
Finite element approximation of eigenvalue problems.
*Acta Numer.*, 19:1–120.

Boffi, D., Brezzi, F., and Fortin, M. (2013).
*Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*.
Springer, Heidelberg.

Boffi, D., Brezzi, F., and Gastaldi, L. (2000).
On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form.
*Math. Comp.*, 69(229):121–140.