



Mathematical  
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# Divergence-free discretisations of the Stokes eigenvalue problem

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SIAM UK/IE Meeting, 21st of April 2023

The slide features a decorative background of white geometric shapes on a dark blue background. On the left, there are several individual polygons, including a parallelogram and a diamond. On the right, a large, complex structure is formed by a grid of interconnected polygons, creating a 3D effect of stacked planes.

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# Stokes eigenvalue problem

Find  $(\mathbf{u}, p) \in H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$  such that  $\forall (\mathbf{v}, q) \in H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$ ,

$$\begin{aligned}\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}, p)_{\mathcal{L}^2(\Omega)} &= \lambda_n (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)}, \\ (\nabla \cdot \mathbf{u}, q)_{\mathcal{L}^2(\Omega)} &= 0,\end{aligned}$$

with  $\lambda_n \in \mathbb{C}$ , and  $\nu \in \mathbb{R}_{>0}$  is the fluid viscosity.

## Stokes eigenvalue problem – Laplace form

We introduce the space  $H_{0,0}^1(\Omega) = \{ \mathbf{v} \in H_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \}$ .

Find  $\mathbf{u} \in H_{0,0}^1(\Omega)$  such that  $\forall \mathbf{v} \in H_{0,0}^1(\Omega)$ ,

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} = \lambda_n (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)},$$

with  $\lambda_n \in \mathbb{C}$ , and  $\nu \in \mathbb{R}_{>0}$  is the fluid viscosity.

## Divergence-free discretisations

$$V_h \subset H_0^1(\Omega) \quad Q_h \subset \mathcal{L}^2(\Omega) \quad \nabla \cdot V_h \subset Q_h$$

Under this hypothesis, we have the following result, i.e.

$$b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0 \Leftrightarrow \nabla \cdot \mathbf{v}^h = 0,$$

which implies the functions are point-wise **divergence-free**.

$$\mathbb{K}_h = \left\{ \mathbf{v}_h \in V_h : b(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h \right\} \subset H_{0,0}^1(\Omega)$$

# Divergence discretisations eigenvalue problem

Find  $\mathbf{u}_h \in \mathbb{K}_h$  such that  $\forall \mathbf{v}_h \in \mathbb{K}_h$ ,

$$\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} = \lambda_n^h (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)},$$

with  $\nabla \cdot \mathbf{V}_h \subset Q_h$ ,  $\lambda_n \in \mathbb{C}$ ,  $\nu \in \mathbb{R}_{>0}$  is the fluid viscosity.

**This problem is well-posed and we can analyse it using  
Babuška-Osborn theory.**

# Finite Element Exterior Calculus

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_0^2(\Omega) & \xrightarrow{\nabla \times} & [H_0^1(\Omega)]^2 & \xrightarrow{\nabla \cdot} & \mathcal{L}_0^2(\Omega) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_h & \xrightarrow{\nabla \times} & V_h & \xrightarrow{\nabla \cdot} & Q_h & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_h & \xrightarrow{\nabla \times} & V_h & \xrightarrow{\nabla \cdot} & \hat{Q}_h & \longrightarrow & 0 \end{array}$$

**Thank you for your attention !**