Divergence-free discretisations of the Stokes eigenvalue problem

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Stokes eigenvalue problem

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Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$ such that $\forall (\mathbf{v}, q) \in H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$,

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}, p)_{\mathcal{L}^2(\Omega)} = \lambda_n \, (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)},$$
$$(\nabla \cdot \mathbf{u}, q)_{\mathcal{L}^2(\Omega)} = 0,$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

Stokes eigenvalue problem – Laplace form

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We introduce the space $H^1_{0,0}(\Omega) = \left\{ \mathbf{v} \in H^1_0(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \right\}$.

Find $\mathbf{u} \in H^1_{0,0}(\Omega)$ such that $\forall\, \mathbf{v} \in H^1_{0,0}(\Omega)$,

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} = \lambda_n \, (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)},$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

Divergence-free discretisations

$$V_h \subset H_0^1(\Omega) \qquad Q_h \subset \mathcal{L}^2(\Omega) \qquad \nabla \cdot V_h \subset Q_h$$

Under this hypothesis, we have the following result, i.e.

$$b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0 \Leftrightarrow \nabla \cdot \mathbf{v}^h = 0,$$

which implies the functions are point-wise **divergence-free**.

$$\mathbb{K}_h = \left\{ \mathbf{v}_h \in V_h \ : \ b(\mathbf{v}_h, q_h) = 0, \forall\, q_h \in Q_h \right\} \subset H_{0,0}^1(\Omega)$$

Divergence discretisations eigenvalue problem

Find $\mathbf{u}_h \in \mathbb{K}_h$ such that $\forall\, \mathbf{v}_h \in \mathbb{K}_h$,

$$\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} = \lambda_n^h\, (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)},$$

with $\nabla \cdot V_h \subset Q_h$, $\lambda_n \in \mathbb{C}$, $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

This problem is well-posed and we can analyse it using Babuška-Osborn theory.

Finite Element Exterior Calculus

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_0^2(\Omega) & \xrightarrow{\nabla \times} & \left[H_0^1(\Omega)\right]^2 & \xrightarrow{\nabla \cdot} & \mathcal{L}_0^2(\Omega) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_h & \xrightarrow{\nabla \times} & V_h & \xrightarrow{\nabla \cdot} & Q_h & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S_h & \xrightarrow{\nabla \times} & V_h & \xrightarrow{\nabla \cdot} & \hat{Q}_h & \longrightarrow & 0
\end{array}
$$

Thank you for your attention !