Divergence-free discretisations of the Stokes eigenvalue problem

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https://github.com/UZerbinati/EFEF2023

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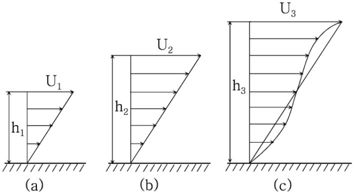
The applications

- The eigenvalues of the Stokes eigenproblem play a crucial role in computing a critical value for the Reynolds number, above which we will predict instabilities in Coutte flow.



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Our contribution

- We realised that the **weak-approximability condition** is automatically verified in divergence-free discretisations, this allows for an easier analysis rather than using the **Boffi–Brezzi–Gastaldi theory**.
- We prove well-posedness **without characterising** the **range** of the **discrete divergence operator**. This characterisation was needed for previous proofs.
- We develop **best approximation estimates**, independent of the **inf-sup**, for functions living in the **kernel** of the **discrete divergence operator**, using **finite element exterior calculus**.

Weak Stokes eigenvalue problem

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Find $(\mathbf{u}, p) \in H^1_0(\Omega) \times \mathcal{L}^2_0(\Omega)$ such that $\forall (\mathbf{v}, q) \in H^1_0(\Omega) \times \mathcal{L}^2_0(\Omega)$,

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}, p)_{\mathcal{L}^2(\Omega)} = \lambda_n \, (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)},$$
$$(\nabla \cdot \mathbf{u}, q)_{\mathcal{L}^2(\Omega)} = 0,$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

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- Are the eigenvalue of this problem real?
- Do the eigenvalue of this problem diverge?

Weak Stokes eigenvalue problem – Laplace form

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We introduce the space $H^1_{0,0}(\Omega) = \left\{ \mathbf{v} \in H^1_0(\Omega) \,:\, \nabla \cdot \mathbf{u} = 0 \right\}$, to obtain an **equivalent** formulation.

Find $\mathbf{u} \in H^1_{0,0}(\Omega)$ such that $\forall\, \mathbf{v} \in H^1_{0,0}(\Omega)$,

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- The eigenvalue problem is self-adjoint therefore $\lambda_n \in \mathbb{R}$.
- $H^1_{0,0}(\Omega)$ is compactly embedded in $\mathcal{L}^2(\Omega)$ and therefore operator corresponding to the eigenvalue problem is compact, implying $\lambda_n \to \infty$ as $n \to \infty$.

Discrete weak Stokes eigenvalue problem

Find $(\mathbf{u}^h, p^h) \in V_h \times Q_h$ such that $\forall\, (\mathbf{v}^h, q^h) \in V_h \times Q_h$,

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Under what hypotheses on V_h and Q_h is this eigenvalue problem well-posed?

Babuška–Banach–Brezzi condition

Inf-Sup

The **necessary** and **sufficient** condition for the well-posedness of the **source** problem is given by the **inf-sup** condition, i.e.

$$\inf_{p_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{(\nabla \cdot v_h, p_h)_{\mathcal{L}^2(\Omega)}}{\|\mathbf{v}_h\|_{H^1(\Omega)} \|p_h\|_{\mathcal{L}^2(\Omega)}} \geq \beta,$$

where β ideally is independent of h.

F. Brezzi and I. Babuška

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Necessary and sufficient conditions

When it comes to the eigenvalue problem of the Stokes type, the inf-sup condition is **not necessary**.

Enio De Giorgi, 1928–1996

Boffi-Brezzi-Gastaldi observation

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Q1-P0 Example

Boffi, Brezzi and Gastaldi, showed that the **Q1-P0** finite element pair will lead to a converging eigenvalue problem, even if for this choice of element pair $\beta(h) \searrow 0$ as $h \to 0$.

Enio De Giorgi, 1928–1996

Boffi–Brezzi–Gastaldi conditions for Stokes

- We say that Q_h verifies the **weak approximability condition** if there exists $\gamma_1(h)$, such that for every $q \in \mathcal{L}^2_0(\Omega)$

$$\sup_{\mathbf{v}^h \in \mathbb{K}_h} \frac{(\nabla \cdot \mathbf{v}^h, q)}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \leq \omega_1(h) \|q\|_{\mathcal{L}^2(\Omega)} \text{ and } \lim_{h \to 0} \gamma_1(h) = 0.$$

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- We say V_h verifies the **strong approximability condition** if there exists $\gamma_2(h)$, such that for every $\mathbf{v} \in H^1_{0,0}(\Omega) \cap H^2(\Omega)$

$$\inf_{\mathbf{v}^h \in \mathbb{K}_h} \left\|\mathbf{v} - \mathbf{v}^h\right\|_{H^1(\Omega)} \leq \gamma_2(h) \|\mathbf{v}\|_{H^2(\Omega)} \text{ and } \lim_{h \to 0} \gamma_2(h) = 0.$$

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$$\boxed{\mathbb{K}_h \not\subset H^1_{0,0}(\Omega)}$$

Divergence-free discretisations

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Under this hypothesis, we have the following result, i.e.

$$b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0 \Leftrightarrow \nabla \cdot \mathbf{v}^h = 0,$$

which implies the functions are point-wise **divergence-free**.

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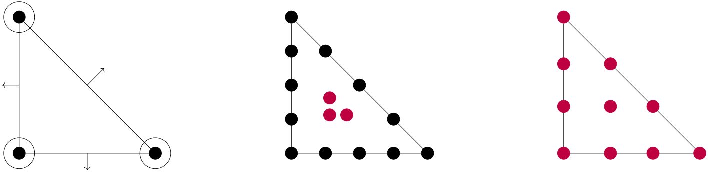
This problem is well-posed and we can analyse it using Babuška-Osborn theory.

Finite Element Exterior Calculus

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_0^2(\Omega) & \xrightarrow{\nabla\times} & \left[H_0^1(\Omega)\right]^2 & \xrightarrow{\nabla\cdot} & \mathcal{L}_0^2(\Omega) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Sigma_h & & \Phi_h & & \Xi_h & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_h & \xrightarrow{\nabla\times} & \Phi_h & \xrightarrow{\nabla\cdot} & \Xi_h^* & \longrightarrow & 0
\end{array}$$

A few more example

- $[\mathbb{P}^4(\mathcal{T}_h)]^2 - \mathbb{P}^3_{disc}(\mathcal{T}_h)$, will be a converging scheme on a criss-cross mesh even if this choice of the element is not inf-sup stable. Best approximation estimates can be derived from the Morgan-Scott-Vogelius complex.



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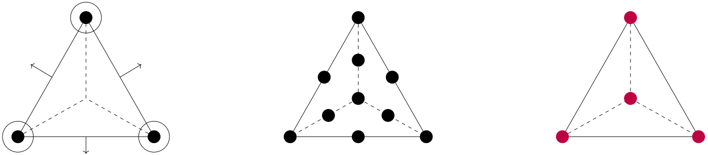
- $[\mathbb{P}^2(\mathcal{T}_h)]^2 - \mathbb{P}^2_{disc}(\mathcal{T}_h)$, will be a converging scheme on a barycentrically refined mesh even if this choice of the element is not inf-sup stable. Best approximation estimates can be derived from Hsieh-Clough-Tocher complex.



Conclusions

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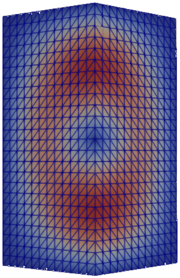

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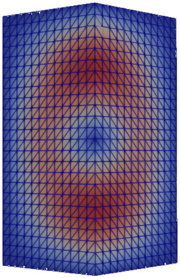

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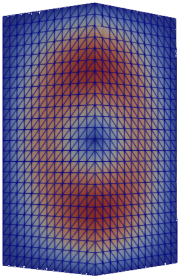

Thank you for your attention!