A Priori Error Analysis For A Penalty Finite Element Method

Thesis by

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Contents

Introduction

Partial differential equations on domains presenting point singularities have always been of interest for applied mathematicians; this interest stems from the difficulty to prove regularity results for non-smooth domains, which have important consequences in the numerical solution of partial differential equations. In my thesis I address those consequences on a particular family of numerical schemes, known as penalty finite element methods. In particular the aim of my thesis is not only to introduce the penalty finite element methods and their a priori error analysis but also to provide a priori error estimates that show that under suitable conditions penalty finite element methods are not inferior to conforming finite element methods. I am also going to show numerical evidence that the penalty finite element methods outperform conforming finite elements method in domains presenting a corner singularity provided that we choose the correct penalisation factor. I would like to keep my introduction short and just provide the reader with an account of the contents of the various chapters in order to allow for an "on-demand" reading. In the first chapter of my thesis I address the continuos problem on singular domains, in particular I limited my self to the Poisson equation. In the first section I address the connection between the Poisson equation with different boundary conditions and energy minimization principles, in particular I describe R. Courant's point of view which will serve as a moral foundation for the penalty finite element method. In the next section I introduce the notion of Sobolev spaces and various existence and uniqueness results in the Sobolev space context; furthermore I will shown different flavours of proof for these results and address some facts that might concern a reader more familiar with well known results in functional analysis. In the last section I introduce the reader to classical results in regularity theory of elliptic partial differential equations and to the seminal work by P. Grisvard on elliptic regularity in non smooth domains. I conclude with detailed computations regarding a specific Pacman like domain, that will accompany the reader throughout the entirety of my thesis in order to make the idea presented clearer. In the second chapter of my thesis I will introduce the reader to the notion of Muckenhoupt weighted Sobolev spaces, I will then recast the seminal work by V. Kondratiev and V. G. Maz'ya in the framework of Muckenhoupt weighted Sobolev spaces and take advantage of this connection in order to show the existence of a variety of Poincaré type inequalities for Maz'ya-Sobolev spaces with different metrics. Last I will present an interpolant developed by R. Nochetto and I will limit myself to the case of piece-wise linear polynomials. I will later on make use of the connection between Maz'ya-Sobolev and Muckenhoupt weighted Sobolev spaces to discuss the approximation of functions in Maz'ya-Sobolev spaces by piece-wise linear polynomials. In the last chapter I will begin giving an overview of different finite element methods that can be used to overcome the lack in regularity cased by the non smoothness of the domain we are working with. Next I will discuss in more detail the a priori error estimates for conforming finite element methods in the context of domains presenting point singularities. I will later introduce the reader to penalty finite element methods and apply techniques that have been used in the context of smooth domains but with singular data in order to show a priori error estimates for penalty finite element in point singular domains. Finally, I'm going to redirect the reader to some numerical results presented in the Appendix and discuss the impossibility to prove an a priori error estimate that explains the super convergence observed by means of a duality argument even if such duality argument is carried out in Muckenhoupt weighted Sobolev spaces. Last I will present to the reader a Petrov-Galerkin analysis that allows to prove a priori error estimate in the norm associated with the Maz'ya-Sobolev spaces extending the usual results to penalty finite element.

1

The Poisson Equation

Throughout my thesis I will focus my attention on a single partial differential equation (PDE), the Poisson equation. The reasons that brought to this decision are the fact that the Poisson equation can be considered as a representative “toy” problem within the class of elliptic partial differential equations with constants coefficients and in my opinion the ideas that I will here present will result clearer to the reader if only the Poisson equation is considered.
Given a domain $\Omega \subset \mathbb{R}^d$ and a smooth enough data $f : \Omega \to \mathbb{R}$ the Poisson equation, consists in finding a smooth enough function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = -\sum_{i=1}^{d} \partial^2_{x_i} u = f \text{ in } \Omega \text{ [1]}. \tag{1.1}$$

Later in this chapter we will discuss in more detail the notion of “enough” smoothness and what does this entail for the meaning of the above equality sign. To begin our study of the Poisson equation we assume that f is a twice differentiable function with compact support, i.e. $f \in C^2_c(\Omega)$, therefore it seems natural to ask for u to be at least twice differentiable, i.e. $u \in C^2(\Omega)$. In this context it is clear what equation (1.1) means, u is a twice differentiable function such that for all x in Ω the trace of the Hessian of u evaluated at x is equal to the value of the data f at the point x. The first question that comes to mind when dealing with any kind of equation, is what are the necessary and sufficient conditions in order for a solution to exists and be unique ? In other words is the problem of finding a solution for equation (1.1) well posed ? Indeed if we assume that $\Omega = \mathbb{R}^d$ then equation (1.1) is well posed.

Theorem 0.1. *Let $f \in C^2_c(\mathbb{R}^d)$, then* (1.1) *has an unique solution $u : \mathbb{R}^d \to \mathbb{R}$, furthermore $u \in C^2(\mathbb{R}^d)$.*

Proof. Providing the reader with a full proof of this result would be out of the scope of my thesis but a fully detailed proof can be found in [47], Chapter 2. □

We will now drop the assumption that Ω is the full $\mathbb{R}^d$ favouring a less restrictive hypothesis, i.e. $\Omega \subset \mathbb{R}^d$ open, bounded and with smooth boundary. If we consider equation (1.1) without any additional hypotheses we can easily find counterexamples to the fact that the Poisson equation is well posed. For instance since constants belong to the kernel of the Laplacian operator, if u is a solution of (1.1) then any function of the form

[1] We follow [47] convention to use the minus sign in front of the Laplacian operator.

$$v : \Omega \to \mathbb{R}$$
$$x \mapsto u(x) + C$$

with $C \in \mathbb{R}$ is still a solution of (1.1). In order to obtain a well posed equation we study the system of equations, that is known as the Dirichlet problem for the Poisson equation, i.e.

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Once the Dirichlet problem has been introduced it is possible to prove that under similar assumptions as in Theorem 0.1, problem (1.2) is well posed.

Theorem 0.2. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with smooth boundary, $f \in C^1(\Omega)$ bounded and $g \in C^0(\partial\Omega)$, then* (1.2) *has a unique solution $u : \Omega \to \mathbb{R}$, furthermore $u \in C^2(\Omega)$.*

Proof. The reader interested in the proof of this result can find it in [54], Chapter 4. □

Remark 0.3. The reader might be tempted to relax furthermore the hypothesis imposed on the data, and this attempt might be successful, in particular we can ask for the data to be bounded and locally Hölder continuous and still retrieve C^2 solution, the proof of such a result can be found in [54], Chapter 4. Nevertheless it will be impossible to take a C^0 data, a counter example can be found in [62], Chapter 3.

1 Calculus of Variations

In this section I will elaborate on the connections between the solution of the Poisson equation and the minimization of energy functionals. In particular I will focus on the problem of equilibrium and on the depiction of rigid boundary conditions as a limiting case of natural boundary conditions. The view point presented here comes from an article by R. Courant that not only considers problem of equilibrium but also of vibration, [31]. References for a reader interested in a more detailed account of the ideas presented in this section are [47] and [54]. Let us begin by considering the homogeneous Dirichlet problem, i.e.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In the previous section I redirected the reader interested in a proof of Theorem 0.2 to [54], but now I would like to elaborate on a particular proof for the uniqueness of the solution of the above equation, known as the **energy method**. Let's assume that u_1, u_2 are two solutions of (1.3), then by linearity we know that also the difference between the two solutions is a solution of (1.3), in particular if we define $\delta := u_1 - u_2$ then δ is a solution of (1.3):

$$\begin{cases} -\Delta\delta = 0 \ in \ \Omega, \\ \delta = 0 \ on \ \partial\Omega. \end{cases}$$

If one multiplies the first equation above by $\frac{1}{2}\delta$ and integrates by parts, the resulting equations will give us uniqueness,

$$-\frac{1}{2}\int_{\Omega} \delta\Delta\delta \, d\mathbf{x} = \frac{1}{2}\int_{\Omega} |\nabla\delta|^2 \, d\mathbf{x} = 0. \tag{1.4}$$

In fact we know from Theorem 0.2 that $\nabla\delta : \Omega \to \mathbb{R}$ is a continuous function and therefore (1.4) give us $\nabla\delta \equiv 0$. Last if we combine the fact that δ has null gradient with the fact that δ is null along the boundary we get $\delta \equiv 0$, which means our two solutions u_1, u_2 of (1.3) are identical everywhere. As previously mentioned this method to prove uniqueness is known as the energy method, because the quantity obtained in (1.4) is known as the **Dirichlet energy** of $\delta : \Omega \to \mathbb{R}$, and will be here called $I(\delta)$,

$$\begin{aligned} I &: C_c^1(\Omega) \to \mathbb{R}, \\ u &\mapsto \frac{1}{2}\int_{\Omega} |\nabla u|^2 d\mathbf{x}. \end{aligned}$$

In the remaining part of this section I will elaborate on some of the properties of the Dirichlet energy functional I and related energy functionals. In particular I will begin exploring the connections between the minimization of the Dirichlet energy functional and the solution of (1.3).

Theorem 1.1. *The solution of* (1.3) *is the minimizers of the Dirichlet energy functional in* $C_c^2(\Omega)$*. Moreover the solution of* (1.2) *is the minimizer of the generalised Dirichlet energy functional, i.e.*

$$J_0(\cdot, f) = \frac{1}{2}\int_{\Omega} |\nabla \cdot|^2 \, d\mathbf{x} - \int_{\Omega} f \cdot \, d\mathbf{x}, \tag{1.5}$$

in $C_C^2(\Omega)$*, provided enough smoothness is assumed on* $f : \Omega \to \mathbb{R}$*, in particular we will here work under the assumption that* $f \in C^1(\Omega)$*.*

Proof. Let us consider a generic $v \in C_c^1(\Omega)$, multiply the first equation in (1.3) by $(v - u)$ and integrate by parts to obtain the following expression:

$$\begin{aligned} -\int_{\Omega} (v-u)\Delta u \, d\mathbf{x} = -\int_{\Omega} v\Delta u - u\Delta u \, d\mathbf{x} = \int_{\Omega} \nabla v \nabla u \, d\mathbf{x} - \int |\nabla u|^2 \, d\mathbf{x} = 0, \\ \frac{1}{2}\int_{\Omega} |\nabla v|^2 \, d\mathbf{x} + \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, d\mathbf{x} - \int_{\Omega} |\nabla u|^2 \, d\mathbf{x} \geq 0, \\ \frac{1}{2}\int_{\Omega} |\nabla v|^2 \, d\mathbf{x} \geq \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, d\mathbf{x}, \end{aligned} \tag{1.6}$$

where Yang's inequality has been used to pass from the first to the second equation. Now since we can obtain an expression as (1.6) for any $v \in C_C^1(\Omega)$ then we have proven that the solution of the homogeneous Dirichlet problem is also a minimizer of the Dirichlet energy functional. If we start from equation (1.2) rather then (1.3) and we proceed as above, we obtain the following:

$$-\int_\Omega (v-u)\Delta u \; d\mathbf{x} = -\int_\Omega v\Delta u - u\Delta u \; d\mathbf{x} = \int_\Omega \nabla v \nabla u \; d\mathbf{x} - \int |\nabla u|^2 \; d\mathbf{x} = \int_\Omega (v-u) f \; d\mathbf{x},$$
$$\frac{1}{2}\int_\Omega |\nabla v|^2 \; d\mathbf{x} + \frac{1}{2}\int_\Omega |\nabla u|^2 \; d\mathbf{x} - \int_\Omega |\nabla u|^2 \; d\mathbf{x} \geq \int_\Omega (v-u) f \; d\mathbf{x},$$
$$\frac{1}{2}\int_\Omega |\nabla v|^2 \; d\mathbf{x} - \int_\Omega fv \; d\mathbf{x} \geq \frac{1}{2}\int_\Omega |\nabla u|^2 \; d\mathbf{x} - \int_\Omega fu \; d\mathbf{x},$$

□

Theorem 1.2. *The critical points of the generalised Dirichlet energy functional in $C_c^2(\Omega)$ are solutions of* (1.2).

Proof. Let u be the minimizer of (1.5) and $v \in C_c^\infty(\Omega)$ then we introduce the following auxiliary function:

$$i : \mathbb{R} \to \mathbb{R}$$
$$\tau \mapsto J_0(u+\tau v, f) = \frac{1}{2}\int_\Omega |\nabla(u+\tau v)|^2 \; d\mathbf{x} - \int_\Omega f(u+\tau v) \; d\mathbf{x}$$

We notice that since u is the minimizer of (1.5) then it is also a critical point for τ and therefore $i'(0) = 0$. Computing explicitly the derivative of $i'(\tau)$ we get,

$$i'(\tau) = \frac{1}{2}\int_\Omega 2(\nabla u \cdot \nabla v) d\mathbf{x} - \int_\Omega fv \; d\mathbf{x},$$

therefore from $i'(0) = 0$ and integrating by parts we get,

$$\int_\Omega \nabla u \nabla v \; d\mathbf{x} - \int_\Omega fv \; d\mathbf{x} = -\int_\Omega \Delta u v \; d\mathbf{x} - \int_\Omega fv \; d\mathbf{x} = \int_\Omega (-\Delta u - f) v d\mathbf{x} = 0. \tag{1.7}$$

We conclude applying the fundamental lemma of calculus of variations and observing that the boundary conditions are verified because we searched for the minimizers u in $C_c^2(\Omega)$. □

I would like to draw the reader attention to the fact that in Theorem 1.2 we found that (1.2) are precisely the Euler-Lagrange equations associated with the Lagrangian,

$$\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$
$$(\mathbf{x}, u, \nabla u) \mapsto |\nabla u|^2.$$

In fact in the proof of Theorem 1.2 we showed that the critical points of the Hamiltonian action for the above Lagrangian are the solution of (1.2), since in this case the Hamiltonian action is precisely (1.5). Computing the second derivative of the Lagrangian the reader can easily find out that the Hamiltonian action is a convex functional, in fact substituting $\mathbf{p} = Du$ and $\mathbf{q} = Dv$ we have

$$\partial^2_{\mathbf{p}}\mathcal{L}(\mathbf{p}) = 2 > 0,$$
$$\mathcal{L}(\theta\mathbf{p} + (1-\theta)\mathbf{q}) \leq \theta\mathcal{L}(\mathbf{p}) + (1-\theta)\mathcal{L}(\mathbf{q}),$$
$$\int_\Omega |\theta\nabla u + (1-\theta)\nabla v|^2 \, d\mathbf{x} \leq \frac{\theta}{2}\int_\Omega |\nabla u|^2 \, d\mathbf{x} + \frac{1-\theta}{2}\int_\Omega |\nabla v|^2 \, d\mathbf{x},$$
$$J_0\Big(\theta u + (1-\theta)v\Big) \leq \theta J_0(u) + (1-\theta)J_0(v). \tag{1.8}$$

Since the Dirichlet energy is convex the solution to the Euler-Lagrange equations does not only correspond the critical point of the Hamiltonian action but it is the minimizer of the Dirichlet energy. The minimization result just proven fits nicely with the principle of minimum energy and with this principle in mind I would like to introduce a different energy functional,

$$J_\varepsilon(\cdot, f) = \frac{1}{2}\int_\Omega |\nabla\cdot|^2 \, d\mathbf{x} - \int_\Omega f\,\cdot\, \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} |\cdot|^2 \, ds. \tag{1.9}$$

In fact the rational behind this energy functional is instead of imposing some boundary condition in the minimizing class to impose a penalisation term on the boundary. We can easily check that the above energy functional is convex, in fact following the same reasoning as before with $p = u$ and $q = v$ we obtain,

$$\int_{\partial\Omega} |\theta u + (1-\theta)v|^2 \, d\mathbf{x} \leq \frac{\theta}{2}\int_{\partial\Omega} |u|^2 \, d\mathbf{x} + \frac{1-\theta}{2}\int_{\partial\Omega} |v|^2 \, d\mathbf{x},$$

combining this last inequality with (1.8) we get that $J_\varepsilon(\cdot)$ is convex,

$$J_\varepsilon\Big(\theta u + (1-\theta)v\Big) \leq \theta J_\varepsilon(u) + (1-\theta)J_\varepsilon(v).$$

and therefore once again finding the critical points of the above Hamiltonian action corresponds to an energy minimization problem. Now given the connection we have seen before between (1.2) and (1.5), it comes natural to wonder if there is a one to one correspondence between the minimizers of (1.9) and the solution of a PDE. To answer this question we introduce the following theorem.

Theorem 1.3. *Given an $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^2$ open set with a Lipschitz boundary the solution of the partial differential equation*

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \partial_n u = -2\varepsilon^{-1}u \text{ on } \partial\Omega,{}^2 \\ \\ \end{cases} \tag{1.10}$$

is the minimizer of (1.9)*, in $C^2(\Omega)$. Viceversa the minimizers of* (1.10) *in $C^2(\Omega)$ are solutions of* (1.10)*.*

Proof. Let us consider $v \in C^2(\Omega)$, then we multiply the first equation of (1.10) by $(v-u)$ and integrate by parts,

$$-\int_\Omega (v-u)\Delta u \, d\mathbf{x} = \int_\Omega f(v-u) \, d\mathbf{x}$$
$$\int_\Omega \nabla u \nabla v \, d\mathbf{x} - \int_{\partial\Omega} v\partial_n u \, ds - \int_\Omega \nabla u \nabla u \, d\mathbf{x} + \int_{\partial\Omega} u\partial_n u \, ds = \int_\Omega fv - \int_\Omega fu$$

now using the second equation in (1.10) and Yang's inequality we get,

$$\int_\Omega \nabla u \nabla v \, d\mathbf{x} + 2\varepsilon^{-1}\int_{\partial\Omega} uv \, ds - \int_\Omega \nabla u \nabla u \, d\mathbf{x} + 2\varepsilon^{-1}\int_{\partial\Omega} |u|^2 \, ds = \int_\Omega fv - \int_\Omega fu,$$
$$\frac{1}{2}\int_\Omega \nabla v \nabla v \, d\mathbf{x} + \frac{1}{2}\int_\Omega \nabla u \nabla u \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} |u|^2 \, ds + \varepsilon^{-1}\int_{\partial\Omega} |v|^2 \, ds - \int_\Omega \nabla u \nabla u \, d\mathbf{x}$$
$$-2\varepsilon^{-1}\int_{\partial\Omega} |u|^2 \, ds \geq \int_\Omega fv - \int_\Omega fu,$$
$$\frac{1}{2}\int_\Omega |\nabla v|^2 \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} |v|^2 \, ds - \int_\Omega fv \, d\mathbf{x} \geq \frac{1}{2}\int_\Omega |\nabla u|^2 \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} |u|^2 \, ds - \int_\Omega fu \, d\mathbf{x}.$$

In order to prove the second part of this theorem we need to take a slightly more complicated route then the one shown in Theorem 1.2. We begin as usual fixing $v \in C^\infty(\Omega)$ and introducing the auxiliary functional,

$$j : \mathbb{R} \to \mathbb{R},$$
$$\tau \mapsto J_\varepsilon(u + \tau v),$$
$$\tau \mapsto \frac{1}{2}\int_\Omega |\nabla u + t\nabla v|^2 \, d\mathbf{x} - \int_\Omega fu + tfv \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} |u + tv|^2 \, d\mathbf{x}.$$

Since we assumed that u is a minimizer of (1.9) we know that $j'(0) = 0$, i.e.

$$j'(t) = \int_\Omega \nabla u \nabla v \, d\mathbf{x} + \int_\Omega |\nabla u|^2 \, d\mathbf{x} - \int_\Omega fv \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} 2tv^2 \, ds + \varepsilon^{-1}\int_{\partial\Omega} 2vu \, ds$$
$$j'(0) = \int_\Omega \nabla u \nabla v \, d\mathbf{x} - \int_\Omega fv \, d\mathbf{x} + 2\varepsilon^{-1}\int_{\partial\Omega} vu \, ds = 0 \qquad \forall v \in C^\infty(\Omega). \tag{1.11}$$

Now we fix any $\omega \subset \Omega$ and consider $v \in C_c^\infty(\omega)$, then the above expression becomes,

$$\int_\omega \nabla u \nabla v \, d\mathbf{x} - \int_\omega fv \, d\mathbf{x} \qquad \forall v \in C^\infty(\omega),$$

integrating by parts we arrive at,

[2] I would like to warn the reader interested in the original view point by R. Courant, presented in [31], that given the fact we adopted the convection to consider the negative Laplacian, as in [47], also the boundary conditions have the been considered with a negative sign.

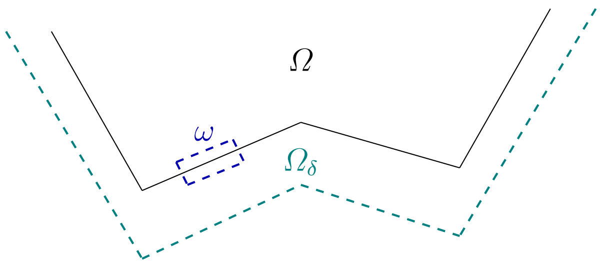
$$-\int_{\omega} v\Delta u \, d\mathbf{x} - \int_{\omega} fv \, d\mathbf{x} \qquad \forall v \in C^{\infty}(\omega).$$

From the previous expression varying arbitrarily $\omega \subset \Omega$ and using the fundamental lemma of calculus of variations we get that the minimizer of (1.9) solves $-\Delta u = f$ in Ω. Therefore we are left to deal with the boundary part of our energy functional which after integrating by parts (1.11) becomes,

$$\int_{\partial\Omega} v\partial_n u \, ds + 2\varepsilon^{-1}\int_{\partial\Omega} vu \, ds = \int_{\partial\Omega} \left(\partial_n u + 2\varepsilon^{-1}u\right)v = 0 \qquad \forall v \in C^{\infty}(\Omega). \tag{1.12}$$

In order to address this problem we extend the domain Ω to a larger domain Ω_δ and consider an open set $\omega \subset \Omega_\delta$ that envelops a smooth portion of the boundary, Figure 1.1. We will now

Fig. 1.1: The below figure depicts the idea behind the tubular neighbourhood variation in $\mathbb{R}^2$.



consider an extension u_δ of u on Ω_δ which is identical to $u_{|\partial\Omega\cap\omega}$ on ω and observe that from (1.12) it follows that,

$$\int_{\partial\Omega} v\partial_n u_\delta \, ds + 2\varepsilon^{-1}\int_{\partial\Omega} vu_\delta \, ds = \int_{\partial\Omega} \left(\partial_n u_\delta + 2\varepsilon^{-1}u_\delta\right)v = 0 \qquad \forall v \in C_c^{\infty}(\omega).$$

The above expression allows us to use once again the fundamental lemma of calculus of variation in order to obtain $\partial_n u_\delta = -2\varepsilon^{-1}u_\delta$ on ω and therefore on $\partial_n u = -2\varepsilon^{-1}u$ on $\partial\Omega \cap \omega$. Last we observe that since the singularity has measure zero with respect to the boundary, then varying ω arbitrarily along the boundary yields $\partial_n u = -2\varepsilon^{-1}u$ on $\partial\Omega$. □

Remark 1.4. I would like to warn the readers to proceed with caution when dealing with the above proof, in fact there are many technicals detail hidden for clarity. One among all is the construction of the extension u_δ, which might seem simple on a convex polygonal domain in $\mathbb{R}^2$ but gets more delicate when dealing with a Liptshiz boundary in $\mathbb{R}^2$. Discussing this in more detail would go outside of the scope of my thesis, but I would like to mention that when dealing with domains that have piecewise smooth boundary if u enjoys a radial symmetry with respect to the singularity point along the boundary then u_δ can still be constructed quite easily. In particular we will deal later on with domain and u as the one previously described. A more general result on the construction of u_δ is the tubular neighbourhood Theorem; more detail can be found in [36], Chapter 2.

We now have two different PDEs, (1.10) which represent a natural constrain and (1.3) which represents a rigid constraint. Furthermore the energy corresponding with (1.3), i.e. (1.5), is the limit as ε approaches zero of the energy corresponding to (1.10), i.e. (1.9). In particular this suggests that we can view the rigid constraint as the limit of natural constraint, as the restoring force of the constraint approaches infinity. This corresponds to the physical idea behind a rigid constraint. A careful reader at this point might be bothered, and with reasons, by the vague concept of limit that has been used in this last paragraph; let me reassure this careful reader that after introducing the notion of weak solution for the two problems, we will show that the solution u_0 to problem (1.3) is the limit of u_ε, the solution of (1.10), as $\varepsilon \to 0$, with respect to a certain norm.

Remark 1.5. An even more careful reader might feel deceived that while at the beginning of this paragraph we try to convey the idea of convergence of energy functionals, we then plan to formalize this notion by the converge of the energy minimizers rather then in terms of the functionals itself. I will address this observations in a later section towards the end of this chapter.

2 Sobolev Spaces and Weak Formulation

Until this section we have always considered the Dirichlet problem with smooth data, in particular we started assuming $f \in C_c^2(\Omega)$ in Theorem 0.1 and relaxed this assumption to $f \in C^1(\Omega)$ in Theorem 0.2. Nevertheless, as already discussed in Remark 0.3, it is not possible to decrease the regularity further, for example $f \in C^0(\Omega)$, and still have $u \in C^2(\Omega)$. As usually done in mathematics if the hypothesis of a Theorem are too stringent for the thesis the only thing left to do is to relax the thesis. In particular in this section we will give a weaker concept of solution for (1.2) which will yield a less regular solution, therefore allowing for a less smooth data. We first observe that if $u \notin C^2(\Omega)$ then we can not give sense to the classical definition of the Laplacian, therefore we need to generalize somehow the notion of second derivative, in order to do so I will first introduce the concept of distribution and then the one of distributional derivative. I reassure the reader that the notion of weak solution and of classical solution will be reconciled in a final remark once all the notion needed to do so will be introduced.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set then a* ***distribution*** *is a linear map, $T : C_c^\infty(\Omega) \to \mathbb{R}$ such that for any $\{\varphi_n\}_{n\in\mathbb{N}}$ converging to $\varphi \in C_c^\infty(\Omega)$ then $\lim_{n\to\infty} T\varphi_n = T\varphi$. We denote the space of all distribution defined on the domain Ω as $D'(\Omega)$.*

Remark 2.2. In particular given a locally integrable function, i.e. $f \in \mathcal{L}^1_{loc}(\Omega)$, there is a canonical correspondence between $\mathcal{L}^1_{loc}(\Omega)$ and $D'(\Omega)$, i.e.

$$\mathcal{L}^1_{loc} \to D'(\Omega)$$
$$f \mapsto T_f$$

where the operator T_f is defined by its action, $< T_f, \varphi >:= \int_\Omega f\varphi \, d\mathbf{x}$. The same identification will hold also for functions in $\mathcal{L}^p(\Omega)$, since $\mathcal{L}^p(\Omega) \subset \mathcal{L}^1_{loc}(\Omega)$, for all $p \in [1, \infty]$.

Definition 2.3. *Given a distribution it is always possible to define a* ***distributional derivative,*** *in particular given $T \in D'(\Omega)$ then $D_i T \in D'(\Omega)$ is called the partial distributional derivative in the x_i direction if $< T, \partial_{x_i}\varphi >= - < D_i T, \varphi >$ for all $\varphi \in C_c^\infty(\Omega)$.*

For the reader interested in developing a deeper insight in the theory of distributions I suggest [49], [82] and [90]. Now that we have a weak enough concept of derivative I would like to do some symbolic calculation in order to see what would be a good definition of weak solution. We begin considering $v \in C_c^\infty(\Omega)$, from now on called **test function**, then we multiply (1.2) by v and integrate by parts to obtain,

$$
\begin{aligned}
-\int_\Omega v \Delta u \, d\mathbf{x} = -\int_\Omega \sum_{i=1}^d v \, \partial_{x_i}^2 u \, d\mathbf{x} = \int_\Omega \sum_{i=1}^d \partial_{x_i} v \, \partial x_i u \, d\mathbf{x} \\
\sum_{i=1}^d \int_\Omega \partial_{x_i} v \, \partial x_i u \, d\mathbf{x} = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \, d\mathbf{x}.
\end{aligned} \tag{1.13}
$$

in order to give sense to (1.13) for solutions that are not twice differentiable we can use the notion of distributional derivative , i.e.

$$
\sum_{i=1}^d < D_{x_i} T_u, \partial_{x_i} v >= \int_\Omega f v \, d\mathbf{x}, \qquad \forall v \in C_c^\infty(\Omega). \tag{1.14}
$$

At this point reader might be bothered by the miss match in regularity between the space where the solution live and the space where we take the test functions, in fact while we take the test functions in $C_c^\infty(\Omega)$, it is enough that T_u is a distribution whose derivative can be represented as an $\mathcal{L}^1_{loc}(\Omega)$ function. In order to deal which this missmatch in regularity we introduce a powerful concept, the idea of Sobolev spaces.

Definition 2.4. *Let Ω be an open set in $\mathbb{R}^d$ and consider $u \in \mathcal{L}^p(\Omega)$, then by virtue of Remark 2.2 we find the corresponding canonical distribution $T_u \in D'(\Omega)$,*

$$
\begin{aligned}
T_u : C_c^\infty(\Omega) \to \mathbb{R} \\
\varphi \mapsto \int_\Omega u \varphi \, d\mathbf{x}.
\end{aligned}
$$

If T_u admits a partial distributional derivative $D_{x_i} T_u$, in all directions, and furthermore it exist $Dx_i u \in \mathcal{L}^p(\Omega)$ such that the following statement holds,

$$
\begin{aligned}
D_{x_i} T_u : C_c^\infty(\Omega) \to \mathbb{R} \\
\varphi \mapsto \int_\Omega D_{x_i} u \, \varphi \, d\mathbf{x}
\end{aligned}
$$

then we say that u lives in the Sobolev space $\mathcal{W}^{1,p}$. Furthermore if for $i \in \{1, \dots, d\}$ we have that $D_{x_i} u$ lives in $\mathcal{W}^{1,p}$ then u lives in $\mathcal{W}^{2,p}$, this allows to define inductively the Sobolev space $\mathcal{W}^{s,p}$, for any $s \in \mathbb{N}$. In particular in the case $p = 2$ it is usual to denote the Sobolev space $\mathcal{W}^{s,2}$ as $H^s(\Omega)$.

The plan of action for the reminder of this section will be as follow, first I will give a brief overview of some useful properties of Sobolev spaces, next I will introduce the notion of weak solution for (1.2), then I will prove some useful result to show existence and uniqueness of weak solutions. I redirect the reader interested in getting a deeper knowledge regarding the topics treated in the remainder of this section to [47, 82, 22, 70, 71, 72, 35].

Proposition 2.5. *The Sobolev space $\mathcal{W}^{s,p}(\Omega)$ is a Banach space with respect to the following norm,*

$$\|u\|^p_{\mathcal{W}^{s,p}(\Omega)} = \|u\|^p_{\mathcal{L}^p(\Omega)} + \sum_{0<|\alpha|\leq s} \|D_\alpha u\|^p_{\mathcal{L}^p(\Omega)},$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ is a multi-index and D_α is the partial distributional derivative taken α_i times in the i-th direction for all $i \in \{1, \dots, d\}$. Furthermore if we select $p = 2$ the space $H^s(\Omega)$ is a Hilbert space with respect to the scalar product,

$$(u, v)_{H^s(\Omega)} = (u, v)_{\mathcal{L}^2(\Omega)} + \sum_{0<|\alpha|\leq s} (D_\alpha u, D_\alpha v)_{\mathcal{L}^2(\Omega)},$$

which induces the above norm for $p = 2$. Last but not least for $p \in (1, \infty)$ the Sobolev space $\mathcal{W}^{s,p}(\Omega)$ is reflexive and for $p \in [1, \infty)$ the space $\mathcal{W}^{s,p}(\Omega)$ is separable.

Proof. I redirect the reader interested in the proof of this result to [22], Chapter 9. □

Proposition 2.6. *Given an open set $\Omega \subset \mathbb{R}^2$ such that $\partial\Omega$ is Lipschitz continuous and $u \in \mathcal{W}^{1,p}$, the trace operator which restricts u on the boundary,*

$$\gamma_0 : \mathcal{W}^{1,p}(\Omega) \to \mathcal{W}^{1-\frac{1}{2},p}(\partial\Omega),$$

is linear and continuous when $p \in [1, \infty)$. Furthermore the trace operator admits a continuous right inverse for $p \in (1, \infty]$. If $d = 2$ then the trace operator,

$$\gamma_0 : H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega),$$

is linear and continuos, for all $n \in \mathbb{N}$.

Proof. The result here synthesised has a long history, that is deeply connected with the University of Pavia, therefore I would like to spend a couple of lines to go through different instances of this result. A classical result from E. Gagliardo states that the trace operator $\gamma_0 : \mathcal{W}^{1,p}(\Omega) \to \mathcal{W}^{1-\frac{1}{2},p}(\partial\Omega)$ is linear and continuous when $p \in [1, \infty)$, furthermore the trace operator admits a continuous right inverse for $p \in (1, \infty]$. The original proof of this result can be found in [50]. The problem for Sobolev space $\mathcal{W}^{s,p}(\Omega)$ with $s > 1$ was addressed later, in particular it was proven by J. Necas that $(\gamma_0, \gamma_1) : \mathcal{W}^{2,p}(\Omega) \to \mathcal{W}^{1,p}(\partial\Omega) \times \mathcal{L}^p(\partial\Omega)$ is a linear and continuos mapping. Characterizations of the range of the above mentioned operator have been developed in two dimension by P. Grisvard in [55], Chapter 3, and then extended to three dimension by A. Buffa and G. Geymonat [23]. Last the result as here stated follows from the proof presented in [52] and the argument in [41]. An account of all this results can be found in [51] □

Definition 2.7. *Given the Sobolev space $\mathcal{W}^{s,p}(\Omega)$ we define the closure of $C^\infty_c(\Omega)$ within this space as $\mathcal{W}^{s,p}_0(\Omega)$, i.e.*

$$\mathcal{W}^{s,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\mathcal{W}^{s,p}(\Omega)}.$$

Another useful notion to introduce at this point is the one of fractional Sobolev space.

Definition 2.8 (Fractional Sobolev Space). *Let $u \in \mathcal{L}^p(\Omega)$, we define the Gagliardo-Slobodeckij seminorm as:*

$$|u|_{\theta,p} := \left(\int_\Omega \int_\Omega \frac{|u(\mathbf{x}) - u(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{\theta p + n}} \, d\mathbf{x} \, d\mathbf{y} \right)^{\frac{1}{p}},$$

where $n = dim(\Omega)$ and $\theta \in (0,1)$. Now it is possible to define the fractional Sobolev space

$$\mathcal{W}^{s,p}(\Omega) = \left\{ u \in \mathcal{W}^{\lfloor s \rfloor,p}(\Omega) \ s.t. \sup_{|\alpha| = \lfloor s \rfloor} |D^\alpha u|_{\theta,p} < \infty \right\},$$

where $s, p \in \mathbb{R}_{>0}$ and $\theta = s - \lfloor s \rfloor$.

I will give for granted some basic properties of fractional Sobolev spaces such as the fact that they are Banach spaces, furthermore when $p = 2$ they are Hilbert spaces if equipped with the following norm

$$\|u\|_{\mathcal{W}^{s,p}(\Omega)} = \|u\|_{\mathcal{W}^{\lfloor s \rfloor,p}(\Omega)} + |u|_{\theta,p}.$$

I redirect the reader interested in getting a deeper knowledge of fractional Sobolev spaces to [35].

2.1 Weak Solutions

We can argue by density to obtain from (1.14) a new notion of solution, i.e. we fix $v \in \mathcal{W}^{1,p}(\Omega)$ and consider a sequence $\{v_n\}_{n \in \mathbb{N}}$ converging to v in $\mathcal{W}^{1,p}(\Omega)$ then (1.14) becomes,

$$\lim_{n \to \infty} \sum_{i=1}^{d} < D_{x_i} T_u, \partial_{x_i} v_n >= \lim_{n \to \infty} \int_\Omega f v_n \, d\mathbf{x}. \tag{1.15}$$

Assuming that the solution of u belongs to the Sobolev space $H^1(\Omega)$, $f \in \mathcal{L}^2(\Omega)$ and the test function was taken in $H^1(\Omega)$ we can swap the limit with the integral in order to get from (1.15):

$$\int_\Omega \sum_{i=1}^{d} D_{x_i} u D_{x_i} v \, d\mathbf{x} = \int_\Omega f v \, d\mathbf{x}.$$

More often then not we will by an abuse of notation write $\nabla u \cdot \nabla v$ to express $\sum_{i=1}^{d} D_{x_i} u D_{x_i} v$.

Definition 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in \mathcal{L}^2(\Omega)$ then we say $u \in H_0^1(\Omega)$ is a weak solution of* (1.3) *if and only if,*

$$a_0(u, v) := \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} = (f, v)_{\mathcal{L}^2(\Omega)}, \qquad \forall v \in H_0^1(\Omega). \tag{1.16}$$

It is easy to see that the above definition is well constructed in the sense that all strong solutions, i.e. solutions of (1.3), are also weak solutions. In fact if we take $v \in H^1_0(\Omega)$, multiply (1.3) by v and integrate by parts we get precisely (1.16). A well known result which will be fundamental when dealing with the bilinear form in (1.16) is Poincare's lemma.

Lemma 2.10 (Poincaré). *] Let Ω be an open bounded set, then there exists a constant $C_P(\Omega)$, greater then zero, such that for all $v \in H^1_0(\Omega)$ we have the following inequality,*

$$\|v\|_{\mathcal{L}^2(\Omega)} \leq C_P(\Omega)\|\nabla v\|_{[\mathcal{L}^2(\Omega)]^d}.$$

Proof. Many different proofs have been developed for this result, I redirect the reader interested in a proof obtained using functional analysis tools to [22], Chapter 9. □

Corollary 2.11. *The bilinear form $a_0(\cdot,\cdot)$ as in Definition 2.9, is coercive in $H^1_0(\Omega)$, i.e. there exists $\alpha > 0$ such that $\forall v \in H^1_0(\Omega)$ the following inequality holds,*

$$\alpha\|v\|_{H^1(\Omega)} \leq a_0(v,v), \qquad \forall v \in H^1_0(\Omega). \tag{1.17}$$

It is possible to connect once again the notion of weak solution with the minimization of the energy functional (1.5).

Proposition 2.12. *Weak solutions of* (1.3)*, defined as in 2.9, are minimizers of* (1.5) *in $H^1_0(\Omega)$, and viceversa.*

Proof. Let us assume that $u \in H^1(\Omega)$ and rewrite $J_0(v)$ in terms of the bilinear form $a_0(\cdot,\cdot)$,

$$\begin{aligned} J_0(v) = \frac{1}{2}a_0(v,v) - (f,v)_{\mathcal{L}^2(\Omega)} =& \frac{1}{2}a_0(u,u) - (f,u)_{\mathcal{L}^2(\Omega)} \\ &+ a_0(u,v-u) - (f,v-u)_{\mathcal{L}^2(\Omega)} \\ &+ \frac{1}{2}a_0(v-u,v-u). \end{aligned}$$

The only thing left to do is to notice that the first line in the previous equation after the equal sign is $J_0(u)$, the second line is null thanks to (1.16) and the last line is greater or equal then zero thanks to the coercivity of $a_0(\cdot,\cdot)$, therefore $J_0(v) \geq J_0(u)$, $\forall v \in H^1_0(\Omega)$. We have already proven the implication the other way around in Theorem 1.2, in particular in (1.7). □

Analogously to what has been done for (1.3), I would like now to define the meaning of weak solution for (1.10) and prove a similar result as in Proposition 2.12.

Definition 2.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in \mathcal{L}^2(\Omega)$ then we say that $u \in H^1(\Omega)$ is a weak solution of* (1.10) *if and only if,*

$$a_\varepsilon(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} + \varepsilon^{-1}\int_{\partial\Omega} vu \, ds = (f,v)_{\mathcal{L}^2(\Omega)}, \qquad \forall v \in H^1(\Omega). \tag{1.18}$$

In order to prove a result similar to Proposition 2.12 we first need to prove coercivity of the bilinear form a_ε.

Lemma 2.14. *The bilinear form $a_\varepsilon(\cdot,\cdot)$ as in Definition 2.13, is coercive, i.e. there exists $\alpha > 0$ such that $\forall v \in H^1(\Omega)$ the following inequality holds,*

$$\alpha \|v\|^2_{H^1(\Omega)} \leq a_\varepsilon(v,v), \qquad \forall v \in H^1(\Omega).$$

Proof. We will prove this by contradiction, therefore we will suppose that $a_\varepsilon(\cdot,\cdot)$ is not coercive, i.e. there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \in H^1(\Omega)$ such that

$$\|v_n\|_{H^1(\Omega)} = 1, \qquad a_\varepsilon(v_n,v_n) = \frac{1}{n} \to 0.$$

Now since $\{v_n\}_{n\in\mathbb{N}}$ is a bounded sequence in the Hilbert space $H^1(\Omega)$ and therefore in a reflexive Banach space, there exists a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ such that, $v_{n_k} \underset{H^1(\Omega)}{\rightharpoonup} v$. Furthermore by Rellich–Kondrachov Theorem we know that, $v_{n_k} \underset{\mathcal{L}^2(\Omega)}{\longrightarrow} v$. Weak convergence in $H^1(\Omega)$ tells us that,

$$\int_\Omega (v - v_{n_k}) w \, d\mathbf{x} + \int_\Omega \nabla(v - v_{n_k}) \cdot \nabla w \, d\mathbf{x} \to 0, \qquad \forall w \in H^1(\Omega).$$

Now if as test function w we take v then we get that,

$$\lim_{k\to\infty} \|\nabla v_{n_k}\|_{\mathcal{L}^2(\Omega)} \|\nabla v\|_{\mathcal{L}^2} \geq \lim_{k\to\infty} \int_\Omega \nabla v_{n_k} \cdot \nabla v \, d\mathbf{x} = \|\nabla v\|^2_{\mathcal{L}^2(\Omega)},$$
$$\lim_{k\to\infty} |v_{n_k}|_{H^1(\Omega)} \geq |v|_{H^1(\Omega)}.$$

Using the fact that $a_\varepsilon(v_n,v_n) = \frac{1}{n} \to 0$ from the above expression we get that,

$$\int_\Omega |\nabla v_n|^2 \, d\mathbf{x} \to 0 \qquad \Rightarrow \qquad \|v\|_{H^1(\Omega)} = \|v\|_{\mathcal{L}^2(\Omega)}.$$

Since $v_{n_k} \underset{\mathcal{L}^2(\Omega)}{\longrightarrow} v$ we get $\|v\|_{\mathcal{L}^2(\Omega)} = \lim_{k\to\infty} v_{n_k} = 1$, and since the gradient of v vanishes we know that $v \equiv \sqrt{\frac{1}{d\mathbf{x}(\Omega)}}$, but this contradicts the fact that $a_\varepsilon(v_n,v_n) = \frac{1}{n} \to 0$ which implies,

$$\int_{\partial\Omega} |v|^2 \, d\mathbf{x} = \lim_{n\to\infty} \int_{\partial\Omega} v_n \, d\mathbf{x} = 0.$$

$\square$

Proposition 2.15. *Weak solutions of* (1.10)*, defined as in 2.13, are minimizers of* (1.9) *in $H^1(\Omega)$, and viceversa.*

Proof. The proof is analogous to the one presented for Proposition 2.12. $\square$

2.2 Existence and Uniqueness

Once the concept of weak derivative has been introduced is time to show that both Definition 2.9 and 2.13 are well posed. The instrument to do this is the **Lax-Milgram** theorem, for which

I would like to give two different proofs. The first one will follow from properties of Hilbert spaces. The second (and more general one) is a consequence of convex minimization results and it is nicely connected to the calculus of variation prospective that we have introduced in the previous section.

Theorem 2.16 (Lax-Milgram). *Let X be a Hilbert space and consider the variational problem,*

$$\text{find } u \in X \text{ such that } a(u,v) = F(v) \qquad \forall v \in X. \tag{1.19}$$

Assuming that the following properties are satisfied:

1. *$a : X \times X \to \mathbb{R}$ is a coercive and continuous bilinear form;*
2. *$F : X \to \mathbb{R}$ is a linear and bounded functional, i.e. $F \in X^*$;*

then the above mentioned variational problem admits a solution, furthermore this solution is unique.

Proof. We notice that if $a(\cdot,\cdot)$ is coercive and continuous then we have the following chain of inequalities holds,

$$\alpha \|\cdot\|_X^2 \leq a(\cdot,\cdot) \leq M\|\cdot\|_X^2,$$

where α and M are respectively the coercivity and the continuity constants. In other words the bilinear form $a(\cdot,\cdot)$ induces a scalar product which is equivalent to the canonical scalar product in X. Therefore using Ritz representation theorem we identify X with its dual X^* and therefore given $F \in X^*$ there exists $u \in X$ such that for all $v \in X$ the following identity holds,

$$a(u,v) = F(v).$$

Now if u_1, u_2 are two solutions of (1.19) then we know that,

$$a(u_1,v) - a(u_2,v) = F(v) - F(v) = 0 \qquad \forall v \in X,$$
$$a(u_1 - u_2, v) = 0 \qquad \forall v \in X,$$

Taking as test function v the difference $u_1 - u_2$ and using the coercivity we get that $\|u_1 - u_2\|_X$ is null, therefore u_1 and u_2 are two identical elements of X. □

Corollary 2.17. *The variational problem* (1.16) *and* (1.18) *have solutions, furthermore the solution in unique.*

Proof. Both variational problem verify the hypothesis of Lax-Milgram theorem, in fact we have already proven the coercivity of a_0 and a_ε, while the continuity of a_ε and a_0 can be obtained simply applying Holder inequality. □

Remark 2.18. I will assume that the reader is as skilled as the writer in Functional Analysis, i.e. very little, and therefore is working with one key principle from F. Brezzi in mind: "Do not identify any space that is not $\mathcal{L}^2(\Omega)$ with its dual", [14] Chapter 4. How can we deal with the fact that we have just violated this principle? Well let us start by observing an instance of where this warning comes from. We know that $H^1(\Omega) \subset \mathcal{L}^2(\Omega)$ and therefore the following embedding exists $\mathcal{L}^2(\Omega)^* \subset H^1(\Omega)^*$, identifying $\mathcal{L}^2$ with its dual we get,

$$H^1(\Omega) \subset \mathcal{L}^2(\Omega) \simeq \mathcal{L}^2(\Omega)^* \subset H^1(\Omega)^*.$$

If we not only identify $\mathcal{L}^2$ with its dual but at the same time we identify $H^1(\Omega)$ with its dual we get the contradiction,

$$H^1(\Omega) \subset \mathcal{L}^2(\Omega) \simeq \mathcal{L}^2(\Omega)^* \subset H^1(\Omega)^* \simeq H^1(\Omega),$$
$$\mathcal{L}^2(\Omega) = H^1(\Omega).$$

This shows us that the problem with identifying a space different from $\mathcal{L}^2(\Omega)$ with its dual, as we do in Theorem 2.16, only subsists if then we also identify $\mathcal{L}^2(\Omega)$ with its dual. This double identification doesn't occur in Theorem 2.16, but might occur later on, therefore we will provide the reader with a proof of Theorem 2.16 that doesn't require the use of Ritz representation theorem.

We will begin introducing the reader to a well known result in convex analysis.

Lemma 2.19. *Given a reflexive Banach space X and a continuous and strongly convex function,*

$$J : X \to \mathbb{R},$$

if the following conditions are satisfied,

1. $\lim_{\|x\| \to \infty} J(x) = +\infty$,
2. *K is a closed convex subset of X,*

then there exists a unique element $x^* \in K$ such that,

$$J(x^*) = \inf_{x \in K} J(x).$$

Proof of Theorem 2.16. First of all we notice that all Hilbert spaces are reflexive Banach spaces, furthermore X being a Hilbert space is convex and closed. Then we consider the energy functional,

$$J : X \to \mathbb{R},$$
$$u \mapsto a(u,u) - F(u).$$

Since $a(\cdot,\cdot)$ is coercive and F is a bounded linear functional, the following chain of inequalities holds,

$$J(u) \geq \alpha \|u\|^2_{H^1(\Omega)} - C_F \|u\|_{H^1(\Omega)}.$$

From the above inequality we clearly see that as $\|u\|_{H^1(\Omega)} \to +\infty$ then $J(u) \to +\infty$. Now in order to apply the previous Lemma 2.19 we only need to check that $J(u)$ is a strictly convex functional. First of all we notice that for all $t \in [0,1]$ and u, v in X we have,

$$
\begin{aligned}
J\Big(tu+(1-t)v\Big) &= a\Big(tu+(1-t)v, tu+(1-t)v\Big) - F\Big(tu+(1-t)v\Big) \\
&= t^2 a(u,u) + 2t(1-t)a(u,v) + (1-t)^2 a(v,v) - tF(u) - (1-t)F(v) \\
&= t^2 a(u,u) - 2t^2 a(u,v) + t^2 a(v,v) + \mathcal{O}(t) \\
&= t^2 a(u-v, u-v) + \mathcal{O}(t),
\end{aligned}
$$

since the coefficient of the quadratic term in $J\Big(tu+(1-t)v\Big)$ is $a(u-v,u-v)$ which is positive given the fact that $a(\cdot,\cdot)$ is coercive, then $J(\cdot)$ is convex. Furthermore since $a(\cdot,\cdot)$ is continuous and coercive the coefficient of the quadratic term is null if and only if $u-v$ is null, i.e. $u=v$. □

Remark 2.20. More often then not the Ritz representation theorem is proven using the reflexivity of Hilbert spaces or by explicitly constructing the Ritz representative. But it is worth mentioning that the Ritz representation Theorem can be viewed as a consequence of Lemma 2.19. I redirect the reader interested in this type of proof to [53].

3 Regularity in Smooth Domains and Point Singular Domains

We would like now to study the regularity of the weak solution for the partial differential equation (1.1). I redirect the reader interested in developing a better understanding of the concepts introduced in this section to [47, 54, 55]. The idea behind elliptic regularity is that ideally the solution u of (1.1) is more regular then its data f. In order to justify this expectations of ours I will begin with a heuristics, i.e. from (1.1) we know

$$
f^2 = (\Delta u)^2,
$$

$$
\int_{\mathbb{R}^d} f^2 d\,\mathbf{x} = \int_{\mathbb{R}^d} (\Delta u)^2 d\,\mathbf{x} = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \partial^2_{x_i} u\, \partial^2_{x_j} u \; d\mathbf{x},
$$

$$
= - \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \partial^2_{x_i} \partial_{x_j} u\, \partial_{x_j} u \; d\mathbf{x} = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \partial_{x_i x_j} u\, \partial_{x_i x_j} u \; d\mathbf{x} = \int_{\mathbb{R}^d} \left|D^2 u\right|^2 \, d\mathbf{x}. \tag{1.20}
$$

From the above computations we get the idea that if the data f lives in $\mathcal{L}^2(\Omega)$ then also $D^2 u$ lives in $\mathcal{L}^2(\Omega)$ and therefore $u \in H^2(\Omega)$. In order to make the above heuristic formal we would need to be able to swap the order of differentiation as done in (1.20); sufficient conditions in order to do so, in terms of C^k regularity, are given by Schwarz Theorem, i.e. $u \in C^3(\Omega)$. Clearly requiring $u \in C^3(\Omega)$ is not useful in order to have $u \in H^2(\Omega)$.

Definition 3.1. *We say that the triplet (A,B,C) is a **shift triplet** for (1.1) if $u \in A$ solution of (1.1) and $f \in B$ imply $u \in C$.*

Theorem 3.2. *Assuming that $a_{ij} \in C^1(\Omega)$ and $b_i, c \in \mathcal{L}^\infty(\Omega)$ then $\Big(H^1(\Omega), \mathcal{L}^2(\Omega), H^2_{loc}(\Omega)\Big)$ is a shift triplet for the weak solution of the Poisson equation in the interior of Ω. Furthermore if ω is compactly included in Ω, i.e. $\omega \subset\subset \Omega$, then we have the following estimate,*

$$
\|u\|_{H^2(\omega)} \le C(\omega,\Omega)\Big(\|f\|_{\mathcal{L}^2(\Omega)} + \|u\|_{\mathcal{L}^2(\Omega)}\Big).
$$

Proof. I redirect the reader interested in the proof of this result to [47], Chapter 6. □

Corollary 3.3. *Let $u \in H^1(\Omega)$ a weak solution of (1.16) with $f \in \mathcal{L}^2(\Omega)$ then u is also a solution (1.1) almost everywhere.*

Proof. Since $\Big(H^1(\Omega), \mathcal{L}^2(\Omega), H^2_{loc}(\Omega)\Big)$ is a shift triplet for (1.16) then we know that for $u \in H^2_{loc}(\Omega)$. Since $u \in H^2_{loc}(\Omega)$ we can integrate by parts to obtain,

$$a(u,v) = \int_\Omega \nabla u \nabla v \; d\mathbf{x} = \int_\Omega v \Delta u \; d\mathbf{x} = \int_\Omega fv \; d\mathbf{x} \qquad \forall v \in C^\infty(\Omega),$$

therefore applying the fundamental lemma of calculus of variation we have that (1.1) is verified almost everywhere. □

Following the same argument that one uses to prove the previous Theorem it is possible to generalize the previous Theorem to higher order derivatives.

Theorem 3.4. *Assuming that $a_{ij} \in C^{m+1}(\Omega)$ and $b_i, c \in \mathcal{L}^\infty(\Omega)$ then $\Big(H^1(\Omega), H^m(\Omega), H^{m+2}_{loc}(\Omega)\Big)$ is a shift theorem for the weak solution of the Poisson equation in the interior of Ω. Furthermore if ω is compactly included in Ω, i.e. $\omega \subset\subset \Omega$, then we have the following estimate,*

$$\|u\|_{H^{m+1}(\omega)} \leq C(\omega,\Omega)\Big(\|f\|_{H^m(\Omega)} + \|u\|_{\mathcal{L}^2(\Omega)}\Big).$$

Until this moment we have considered the regularity elliptic problem without taking into consideration the boundary, this is known as **interior regularity**, it is now time to deal with the **boundary regularity**. I would like to sketch the path the reader can follow in order to use interior regularity to prove boundary regularity. Since Ω is pre compact it is possible to find a finite covering of $\overline{\Omega}$, i.e. $\{\omega_k \subset\subset \Omega \; : \; k \in K\}$. Furthermore if we assume Ω has a C^2 boundary then it is possible to find a sequence of C^2 diffeomorphisms $\Phi_k\big(B_0(r_k) \cap \mathbb{R}^+\big) = \omega_k$. Now we can define the partition of unity ξ_k such that ξ_k is a mollifier with support ξ_k and $\sum_{k\in K} \xi_k = 1$. Last by symmetry we can use the interior regularity, i.e. Theorem 3.2, to assert $\Phi_k(\xi_k u) \in H^2\Big(B_0(r_k) \cap \mathbb{R}^+\Big)$ and reconstruct u as follow,

$$u = \sum_{k\in K} \xi_k u = \sum_{k \in K} \Phi_k^{-1}\Big(\Phi_k\big(\xi_k u\big)\Big) \in H^2(\Omega).$$

Theorem 3.5. *Assuming that $a_{ij} \in C^1(\Omega)$ and $b_i, c \in \mathcal{L}^\infty(\Omega)$ then $\Big(H^1_0(\Omega), \mathcal{L}^2(\Omega), H^2(\Omega)\Big)$ is a shift triplet for (1.16), provided that $\partial\Omega$ is smooth. Furthermore the following estimate holds,*

$$\|u\|_{H^2(\Omega)} \leq C(\omega,\Omega)\Big(\|f\|_{\mathcal{L}^2(\Omega)} + \|u\|_{\mathcal{L}^2(\Omega)}\Big).$$

Corollary 3.6. *Assuming that $a_{ij} \in C^{m+1}(\Omega)$ and $b_i, c \in \mathcal{L}^\infty(\Omega)$ then $\Big(H^1_0(\Omega), H^m(\Omega), H^{m+2}(\Omega)\Big)$ is a shift theorem for (1.16), provided that $\partial\Omega$ is smooth. Furthermore the following estimate holds,*

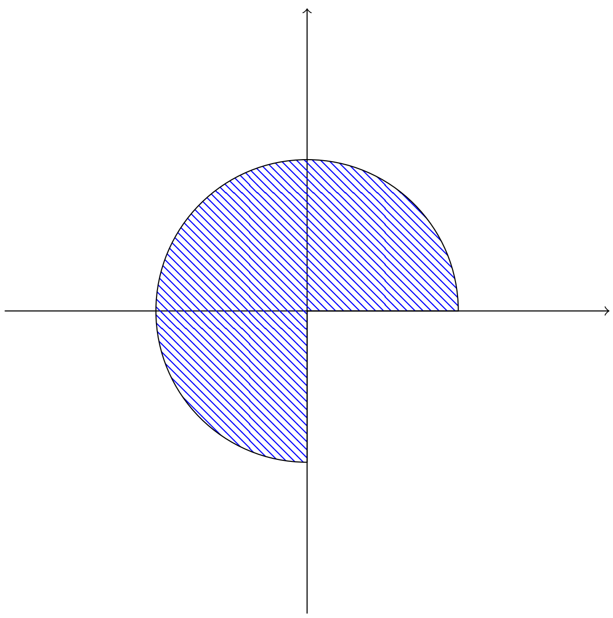
$$\|u\|_{H^{m+2}(\Omega)} \leq C(\omega, \Omega)\Big(\|f\|_{H^m(\Omega)} + \|u\|_{\mathcal{L}^2(\Omega)}\Big).$$

Now a key assumption in order to follow the proof sketch before was that the boundary of Ω was smooth, in particular C^2. If this assumption is dropped things becomes much more complicated. In order to present the regularity of the elliptic problem in domains with corner I would like to introduce the reader to an illuminating example. The idea to use the particular geometry shown in the next example comes from the introduction of [88].

3.1 Pacman Example

In this example we will study the eigenvalue problem associated with (1.3) in the domain depicted in Figure 1.2, using the technique of separation of variables. We will make one further assumption dictated by the physics of the problem, i.e $|\Phi(\rho, \theta)| < \infty$. As usual we assume Φ

Fig. 1.2: In the figure the domain where we solve the Dirichlet eigenvalue problem is drawn.



depends separately upon the radius and the angle of the circular sector, $\Phi(\rho, \theta) = \Theta(\theta)R(\rho)$, expressing the Laplacian in polar coordinates we obtain

$$\Delta\Phi(\rho,\theta) = \partial_x^2\Phi(\rho,\theta) + \partial_y^2\Phi(\rho,\theta) = \frac{1}{\rho}\partial_\rho\Phi(\rho,\theta) + \frac{1}{\rho^2}\partial_\theta^2\Phi(\rho,\theta) + \partial_\rho^2\Phi(\rho,\theta)$$
$$\Delta\Phi(\rho,\theta) = \frac{1}{\rho}R'(\rho)\Theta(\theta) + \frac{1}{\rho^2}R(\rho)\Theta''(\theta) + R''(\rho)\Theta(\theta)$$

therefore imposing the eigenvalue problem we have the following expression,

$$\Delta\Phi(\rho,\theta) = -\lambda\Phi(\rho,\theta)$$
$$\frac{1}{\rho}R'(\rho)\Theta(\theta) + \frac{1}{\rho^2}R(\rho)\Theta''(\theta) + R''(\rho)\Theta(\theta) = -\lambda R(\rho)\Theta(\theta).$$

We perform some algebraic manipulations to obtain on one side an expression in θ and on the other side an expression in ρ.

$$\begin{aligned}
\Big(\frac{1}{R(\rho)\Theta(\theta)}\times\Big) \qquad \frac{1}{\rho}R'(\rho)\Theta(\theta)+\frac{1}{\rho^2}R(\rho)\Theta''(\theta)+R''(\rho)\Theta(\theta) &= -\lambda R(\rho)\Theta(\theta)\\
\Big(\rho^2\times\Big) \qquad \frac{1}{\rho}\frac{R'(\rho)}{R(\rho)}+\frac{1}{\rho^2}\frac{\Theta''(\theta)}{\Theta(\theta)}+\frac{R''(\rho)}{R(\rho)} &= -\lambda\\
\rho\frac{R'(\rho)}{R(\rho)}+\frac{\Theta''(\theta)}{\Theta(\theta)}+\rho^2\frac{R''(\rho)}{R(\rho)} &= -\lambda\rho^2\\
-\frac{\Theta''(\theta)}{\Theta(\theta)} = \rho\frac{R'(\rho)}{R(\rho)}+\rho^2\frac{R''(\rho)}{R(\rho)}&+\lambda\rho^2. \qquad (1.21)
\end{aligned}$$

Now since one side only depend on θ while the other only depend on ρ we obtain a well known one dimensional eigenvalue problem,

$$\begin{cases}\Theta''(\theta) = -\mu\Theta(\theta),\\ \Theta(0)=\Theta(\frac{3}{2}\pi)=0\end{cases}. \qquad (1.22)$$

It is well known that (1.22) has the following solutions,

$$\mu_n = \Big(\frac{2}{3}n\Big)^2 \qquad \Theta(\theta)=\sin\Big(\frac{2}{3}n\theta\Big)$$

which together with (1.21) gives an ODE for $R(\rho)$, i.e.

$$\begin{aligned}
\rho\frac{R'(\rho)}{R(\rho)}+\rho^2\frac{R''(\rho)}{R(\rho)}+\lambda\rho^2-\mu_n&=0,\\
\rho^2R''(\rho)+\rho R'(\rho)+R(\rho)\Big(\lambda\rho^2-\mu_n\Big)&=0.
\end{aligned}$$

Performing the variable change $z=\sqrt{\lambda}\rho$ we obtain Bessel differential equation,

$$z^2R''(z)+zR'(z)+(z^2-\alpha_n^2)R(z)=0, \qquad \alpha_n^2=\mu_n.$$

Bessel ODE has the following solution, $R(z)=AJ_{\alpha_n}(z)+BY_{\alpha_n}(z)$. Since we have the hypothesis $|R(\rho)|<\infty$ as $\rho\to 0$ then $B=0$, i.e.

$$R(r)=J_{\alpha_n}(\sqrt{\lambda}r),$$

using homogeneous Dirichlet boundary condition we get $R(1)=J_{\alpha_n}(\sqrt{\lambda})=0$ and therefore λ must be equal to the square of the m-th zero of J_{α_n}, i.e. $\lambda=(z_{m,n})^2$. This yields the following solution to the eigenvalue problem associated with (1.3),

$$\Phi_{n,m}(\rho,\theta) = J_{\alpha_n}(z_{m,n}\rho)\sin\left(\frac{2}{3}\theta n\right), \qquad\qquad \lambda_{m,n} = z_{m,n}^2.$$

Now we notice that using the fact that when $\rho \to 0$,

$$J_{\alpha_n}(z_{m,n}\rho) \approx \frac{1}{\Gamma(\alpha_n+1)}\left(\frac{z_{m,n}\rho}{2}\right)^{\alpha_n},$$

then we have the following approximation for $\Phi_{n,m}$:

$$\Phi_{n,m}(\rho,\theta) \approx C_{m,n}\rho^{\frac{2}{3}n}\sin\left(\frac{2}{3}n\theta\right). \tag{1.23}$$

Computing the norm in $\mathcal{W}^{1,2}\Big(B_\rho(\mathbf{0})\Big)$ and $\mathcal{W}^{2,2}\Big(B_\rho(\mathbf{0})\Big)$ we can show that $\Phi_{m,1}$ does live in $\mathcal{W}^{1,2}(\Omega)$ but not in $\mathcal{W}^{2,2}(\Omega)$. Furthermore if one uses approximation (1.23) to compute the Gagliardo-Slobodeckij semi-norm we can show that $\Phi_{m,1} \in \mathcal{W}^{\frac{5}{3}-\varepsilon}(\Omega)$. Expanding a generic solution of (1.16) in the eigenspace we notice that the singular functions S_m are of the form $\Phi_{m,1}$ if $f \in \mathcal{L}^2(\Omega)$. In general the following result holds,

Theorem 3.7. *Let us consider a domain $\Omega \subset \mathbb{R}^2$ with a re-entrant corner of aperture ω. If $f \in \mathcal{W}^{0,p}(\Omega)$ then $u_0 \in \mathcal{W}^{1,p}_0(\Omega)$ is such that,*

$$u_0 - \sum_m C_m S_m \in \mathcal{W}^{2,p}(\Omega),$$

where the S_m are a particular set of singular functions belonging to the space $\mathcal{W}^{2-\frac{\pi}{\omega}-\varepsilon,p}(\Omega)$. Furthermore, $\Big(\mathcal{W}^{1,p}_0(\Omega), \mathcal{W}^{0,p}(\Omega), \mathcal{W}^{2,p}(\Omega)\Big)$ is a shift triplet for $u_0 - \sum_m C_m S_m$.

Proof. I redirect the reader interested in the proof of this result to [55], Chapter 4. □

2

Weighted Sobolev Spaces

In this chapter I would like to introduce the notion of weighted Sobolev space, later I will focus my attention to particular classes of weighted Sobolev spaces that are very useful when dealing with singular domains. Last I will show how to approximate functions in weighted Sobolev spaces using a piecewise linear interpolant. The idea of using weighted Sobolev spaces to study singular problems has a long history, I redirect the reader interested in this topic to [39, 37, 42, 12, 34, 78]. Last I redirect the reader interested in generic properties of Muckenhoupt Sobolev spaces to [86] and [2].

Definition 0.1. *A weight is a function $\omega \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$ that is positive almost everywhere.*

One important feature of weights as defined above is that they induce a measure on the Borelian of $\mathbb{R}^d$, i.e.

$$\begin{aligned} \omega : \mathcal{B}(\mathbb{R}^d) &\to \mathbb{R}, \\ E &\mapsto \int_E \omega(\mathbf{x})\, d\mathbf{x}. \end{aligned}$$

It is a well known fact that the above measure is absolutely continuous with respect to the Lebesgue measure, perhaps a more interesting observation is that the Lebsegue measure is absolutely continuos with respect to ω. A class of weights that will be of particular interest to us for the remainder of this chapter are Muckenhoupt weights. This class of weights was first introduced to characterize for which weights is the Hardy-Littlewood operator bounded in $\mathcal{L}^p(\mathbb{R}^d)$, [76]. The reason why we focus on this particular class of weights is that they have some very desirable properties, one among all they have the so called **strong doubling property**, that will be essential in the construction of our interpolant.

Definition 0.2 (Muckenhoupt Weights). *Given a weight $\omega \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$ we say that ω is Muckenhoupt of class p, i.e. $\omega \in A_p(\mathbb{R}^d)$ if there exists $C_{p,\omega} > 0$ such that,*

$$\sup_{\{B : B \text{ ball in } \mathbb{R}^d\}} \left(\fint_B \omega\, d\mathbf{x} \right) \left(\fint_B \omega^{\frac{1}{1-p}}\, d\mathbf{x} \right)^{p-1} = C_{p,\omega} < \infty.$$

I will now introduce the reader to some of the previously mentioned desirable property of Muckenhoupt weights.

Theorem 0.3. *Let $p \in (1, \infty)$, $\omega \in A_p(\mathbb{R}^d)$ then the following statements hold,*

1. $\omega^{\frac{1}{1-p}} \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$*,*
2. $C_{p,\omega} > 1$*,*
3. $A_p(\mathbb{R}^d) \subset A_r(\mathbb{R}^d)$ *and* $C_{p,\omega} \geq C_{r,\omega}$ *for all* p, r *such that* $1 < p \leq r < \infty$*,*
4. $\omega^{\frac{1}{1-p}} \in A_{p'}(\mathbb{A}^d)$ *and* $C_{p',\omega^{\frac{1}{1-p}}} = C_{p,\omega}^{\frac{1}{p-1}}$*, where* $\frac{1}{p} + \frac{1}{p'} = 1$*,*
5. The class of $A_p(\mathbb{R}^d)$ *weights is invariant under the composition of isotropic dilation and translation.*

Proof. Let us prove the above statements in order.

1. From Definition 0.2 we know that for a fixed ball $B \subset \mathbb{R}^d$, we have:

$$\left(\fint_B \omega \, d\mathbf{x}\right)\left(\fint_B \omega^{\frac{1}{1-p}} \, d\mathbf{x}\right)^{p-1} = C_{p,\omega} < \infty,$$

$$\left(\fint_B \omega^{\frac{1}{1-p}} \, d\mathbf{x}\right) = \left(\left(\fint_B \omega \, d\mathbf{x}\right)^{-1} C_{p,\omega}\right)^{\frac{1}{p-1}} < \infty,$$

this is because the Lebesgue measure is absolutely continuous with respect of ω and therefore the quantity $\left(\left(\fint_B \omega \, d\mathbf{x}\right)^{-1}\right.$ doesn't explode.

2. We begin observing that $1 = \omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}$. Using Holder inequality we the obtain,

$$1 = \fint_B \omega^{\frac{1}{p}}\omega^{-\frac{1}{p}} \, d\mathbf{x} \leq \left(\fint_B \omega \, d\mathbf{x}\right)^{\frac{1}{p}} \left(\fint_B \omega^{\frac{1}{1-p}} \, d\mathbf{x}\right)^{\frac{p-1}{p}},$$

the only thing left to do is to evaluate the same expression at the power of p and observe that since we know both terms of the inequality are greater then zero the inequality sign doesn't change order.

3. Once again we make use of Holder inequality in order to observe that if $1 < p \leq r < \infty$ then,

$$\left(\fint_B \omega^{\frac{1}{1-r}} \, d\mathbf{x}\right)^{r-1} \leq \left(\fint_B \omega^{\frac{1}{1-p}} \, d\mathbf{x}\right)^{p-1}.$$

We can multiply the above expression by $\fint_B \omega \, d\mathbf{x}$ and take the supremum on $\{B : B \textit{ ball in } \mathbb{R}^d\}$ to obtain,

$$\sup_{\{B : B \textit{ ball in } \mathbb{R}^d\}} \fint_B \omega \, d\mathbf{x} \left(\fint_B \omega^{\frac{1}{1-r}} \, d\mathbf{x}\right)^{r-1} \leq \sup_{\{B : B \textit{ ball in } \mathbb{R}^d\}} \fint_B \omega \, d\mathbf{x} \left(\fint_B \omega^{\frac{1}{1-}} \, d\mathbf{x}\right)^{p-1}.$$

Now that we know $C_{r,\omega} \leq C_{p,\omega}$ it is clear that all $u \in A_p(\mathbb{R}^d)$ also live in $u \in A_r(\mathbb{R}^d)$, i.e. $A_p(\mathbb{R}^d) \subset A_r(\mathbb{R}^d)$.

4. Applying Definition (0.2) with ω equal to $\omega^{\frac{1}{1-p}}$ we get the desired inequality.
5. We now consider an isotropic dilation composed with a translation $\mathbf{x} \mapsto \alpha\mathbf{x} + \mathbf{b}$ and the weight $\overline{\omega}(\mathbf{x}) = \omega(\alpha\mathbf{x} + \mathbf{b})$, then we notice:

$$\fint_{B_r(\mathbf{x})} \overline{\omega}(\mathbf{x})\, d\mathbf{x} = \fint_{B_r(\mathbf{x})} \omega(\alpha \mathbf{x} + \mathbf{x})\, d\mathbf{x} = \frac{1}{\alpha^n} \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} \omega(\mathbf{y})\, d\mathbf{y} = \fint_{B_{\alpha r}(\mathbf{x}+\mathbf{b})} \omega(\mathbf{y})\, d\mathbf{y},$$

therefore $\overline{\omega}$ is a Muckenhoupt weight, in particular $\overline{\omega} \in A_p(\mathbb{R}^d)$.

□

Corollary 0.4. *Let $\omega \in A_p(\mathbb{R}^d)$ with $p \in (1, \infty)$ and let $E \subset \mathbb{R}^d$ be a measurable subset of $B \subset \mathbb{R}^d$, a ball, then:*

$$\omega(B) \leq C_{p,\omega} \Big(\frac{|B|}{|E|} \Big)^p \omega(E).$$

Proof. Since $E \subset \mathbb{R}$ is measurable then we have,

$$\begin{aligned} |E| = \int_E 1\, d\mathbf{x} = \int_E \omega^{\frac{1}{p}} \omega^{-\frac{1}{p}}\, d\mathbf{x} &\overset{H.}{\leq} \Big(\int_E \omega^{\frac{p'}{p}}\, d\mathbf{x} \Big)^{\frac{1}{p'}} \Big(\int_E \omega\, d\mathbf{x} \Big)^{\frac{1}{p}} \\ &\overset{E \subset B}{\leq} \omega(E)^{\frac{1}{p}} |B|^{\frac{1}{p'}} \Big(\fint_B \omega^{-\frac{p'}{p}}\, d\mathbf{x} \Big)^{\frac{1}{p'}}. \end{aligned} \tag{2.1}$$

We use the fact that ω is a Muckenhoupt which together with $\dfrac{1}{p'} = \dfrac{p-1}{p}$ to obtain,

$$\begin{aligned} \Big(\fint_E \omega\, d\mathbf{x} \Big) \Big(\fint_E \omega^{\frac{1}{1-p}}\, d\mathbf{x} \Big)^{p-1} &\leq C_{p,\omega} < \infty, \\ \Big(\fint_E \omega\, d\mathbf{x} \Big) \Big(\fint_E \omega^{-\frac{p'}{p}}\, d\mathbf{x} \Big)^{p-1} &\leq C_{p,\omega} < \infty, \\ \Big(\fint_E \omega\, d\mathbf{x} \Big)^{\frac{1}{p}} \Big(\fint_E \omega^{-\frac{p'}{p}}\, d\mathbf{x} \Big)^{\frac{1}{p'}} &\leq \Big(C_{p,\omega} \Big)^{\frac{1}{p}} < \infty, \\ \Big(\fint_E \omega^{-\frac{p'}{p}}\, d\mathbf{x} \Big)^{\frac{1}{p'}} &\leq \Big(C_{p,\omega} \Big)^{\frac{1}{p}} \Big(\fint_E \omega\, d\mathbf{x} \Big)^{-\frac{1}{p}} < \infty. \end{aligned}$$

In the last row we didn't change the sign of the inequality because we know from Definition 0.1 the quantity we are interested in is less then one. Furthermore the integral on the right hand side of the above expression is different from zero because we know the Lebesgue measure is absolutely continuous with respect to ω. Combining the last inequality with (2.1) we obtain,

$$|E| \leq \Big(C_{p,\omega} \Big)^{\frac{1}{p}} \omega(E)^{\frac{1}{p}} |B|^{\frac{1}{p'}} \Big(\fint_B \omega\, d\mathbf{x} \Big)^{-\frac{1}{p}} \leq \Big(C_{p,\omega} \Big)^{\frac{1}{p}} \omega(E)^{\frac{1}{p}} |B|^{\frac{1}{p'}} |B|^{\frac{1}{p}} \omega(B)^{-\frac{-1}{p}} \omega(B)^{-\frac{1}{p}},$$

$$|E| \leq \Big(C_{p,\omega} \Big)^{\frac{1}{p}} \left(\frac{\omega(E)}{\omega(B)} \right)^{\frac{1}{p}} |B|.$$

To conclude we just raise everything to the power of p and multiply and divide by what is needed. □

Remark 0.5. A particular case of the above Corollary is the fact that given two balls centred at $\mathbf{x}$, i.e. $B_{2r}(\mathbf{x})$ and $B_r(\mathbf{x})$, we then have:

$$\omega\Big(B_{2r}(\mathbf{x})\Big) \leq C\left(\frac{|B_{2r}(\mathbf{x})|}{B_r(\mathbf{x})}\right)^p,$$
$$\omega\Big(B_{2r}(\mathbf{x})\Big) \leq C\omega\Big(B_r(\mathbf{x})\Big).$$

This last inequality is precisely what we called the **strong doubling property** of the weight $\omega : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$.

Now it is time to introduce the weighted counterpart of the Lebesgue space and of the Sobolev space. Furthermore we will give a brief characterization of those spaces, by some of their major properties.

Definition 0.6. *Given $\omega \in A_p(\mathbb{R}^d)$ and $\Omega \subset \mathbb{R}^d$ a pre-compact set, we define the Muckenhoupt weighted Lebesgue space $\mathcal{L}^p(\omega, \Omega)$ as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that,*

$$\|u\|_{\mathcal{L}^p(\omega,\Omega)} = \Big(\int_\Omega |u|^p\, \omega d\mathbf{x}\Big)^{\frac{1}{p}} < \infty. \tag{2.2}$$

We call $\mathcal{W}^{s,p}(\omega, \Omega)$ the set of measurable $u : \Omega \to \mathbb{R}$ such that $D_\alpha u \in \mathcal{L}^p(\omega, \Omega)$, for any multi-index α such that $|\alpha| \leq s$.

Proposition 0.7. *The Muckenhoupt weighted Lebesgue space $\mathcal{L}^p(\omega, \Omega)$ is a Banach space with respect to the norm defined in* (2.2).

Proposition 0.8. *The Muckenhoupt weighted Lebesgue space $\mathcal{L}^p(\omega, \Omega)$ is a subset of $\mathcal{L}^1_{loc}(\Omega)$.*

Proof. Since ω is a Muckenhoupt weight we know from Theorem 0.3 that $\omega^{-\frac{1}{p-1}} \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$ and therefore for any ball $B \subset \Omega$ the following holds,

$$\int_B |u| = \int_B |u|\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}} \overset{H}{\leq} \Big(\int_B |u|^p\omega\Big)^{\frac{1}{p}}\Big(\int_B \omega^{-\frac{1}{p-1}}\Big)^{\frac{p-1}{p}} < \infty.$$

□

1 Weighted Sobolev Spaces for Singular Domains

For the remainder of this chapter I would like to focus my attention on a particular Muckenhoupt weight, i.e. $\omega(\mathbf{x}) = |\mathbf{x}|^\gamma$. A particular case of weighted Sobolev space with $\omega(\mathbf{x}) = |\mathbf{x}|^\gamma$ are Kondrat'ev and Maz'ya Sobolev spaces on domains with a specific geometry. At this point the reader might be wondering why I want to introduce this "exotic" weighted Sobolev space. The answer is given by the fact that Maz'ya and Kondrat'ev Sobolev spaces allow to retrieve desirable shifts similar to the one presented for smooth domains also in the case of domains with point singularities. In fact Kondrat'ev and Maz'ya Sobolev spaces are a particular case of weighted Sobolev spaces for which the weights depend on the geometry of the domain. I redirect the reader interested in weighted Sobolev space for domains with point singularities to [29, 30, 67, 15].

Proposition 1.1. *The weight* $\omega(\mathbf{x}) = |\mathbf{x}|^\gamma$ *is a Muckenhoupt weight of class* $A_p(\mathbb{R}^d)$ *if* $\gamma \in \Big(-d, d(p-1)\Big)$.

Proof. We begin by observing that given a weight of the form $\mathbf{x} \mapsto |\mathbf{x}|^\gamma$ then we can exactly compute the measure of a ball with respect to ω,

$$\omega\Big(B_r(\mathbf{x_0})\Big) = \int_{B_r(\mathbf{x_0})} \omega(\mathbf{x})\, d\mathbf{x} = \int_{B_r(\mathbf{x_0})} |x|^\gamma\, d\mathbf{x} = \int_{B_r(\mathbf{0})} |\mathbf{y}|^\gamma d\mathbf{y},$$
$$= C\int_0^r |\mathbf{y}|^{\gamma+(d-1)} d|\mathbf{y}| = C\Big|_0^1 r^{\gamma+d},$$

which is finite for $\gamma \in (-n, \infty)$. If we compute the measure of a ball with respect to $\omega^{\frac{1}{1-p}}$, following the same steps presented above, we obtain:

$$\int_{B_r(\mathbf{x_0})} \omega^{\frac{1}{1-p}} = C\Big|_0^1 r^{\frac{\gamma}{1-p}+d},$$

which is finite ig $\gamma < d(p-1)$. Combining this last two computations it is easy to see that if $\omega \in \Big(-d, d(p-1)\Big)$ then ω is a Muckenhoupt weight. $\square$

For the remainder of my thesis I will assume that Ω is a pre-compact subset of $\mathbb{R}^2$ with smooth boundary except for a point where it has a corner of aperture α, similarly to the domain depicted in Figure 1.2. A useful quantity to define is $\beta \in \mathbb{R}$ such that,

$$0 \le -\beta - 1 < \frac{\pi}{\omega}. \tag{2.3}$$

Definition 1.2 (Kondrat'ev-Sobolev Space). *Given a measurable function* $u : \Omega \to \mathbb{R}$ *we define the following quantity,*

$$\|u\|_{\mathcal{K}^m_\beta(\Omega)} = \Big(\sum_{k=0}^m |u|^2_{\mathcal{K}^k_\beta}\Big)^{\frac{1}{2}}, \qquad |u|_{\mathcal{K}^k_\beta(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|^2_{\mathcal{L}^2\left(|\mathbf{x}|^{2\beta+2|\alpha|}, \Omega\right)}\right)^{\frac{1}{2}}.$$

Furthermore we will call the ***Kondrat'ev-Sobolev*** *space,* $\mathcal{K}^m_\beta(\Omega)$*, the set of measurable functions* $u : \Omega \to \mathbb{R}$ *with finite* $\|u\|_{\mathcal{K}^m_\beta}$*. It is important to notice that* β *must be chosen in order to verify* (2.3)*.*

Now it is possible to prove our first shift Theorem, that improves on the one presented at the end ot last chapter, i.e. Theorem 3.7.

Theorem 1.3. *The triplet* $\Big(H^1_0(\Omega), \|u\|_{\mathcal{K}^m_{\beta+2}}(\Omega), \|u\|_{\mathcal{K}^{m+2}_\beta(\Omega)}\Big)$ *is a shift triplet for* (1.16)*.*

Proof. Unfortunately proving this result would be out of the scope of my thesis, but I redirect the reader interested in such result to [15], Chapter 5. $\square$

The limitation of the above stated regularity result, and in particular of Kondrat'ev-Sobolev spaces, is the inability to capture the singularities presented by the eigenfunctions of the mixed Dirichlet-Neumann elliptic problem, i.e.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \Gamma_{\mathrm{N}} \subset \partial\Omega \\ u = 0 & \text{in } \Gamma_{\mathrm{D}} \subset \partial\Omega \end{cases} . \tag{2.4}$$

In order to deal with this particular type of problem we will use Maz'ya-Sobolev spaces, i.e.

Definition 1.4 (Maz'ya-Sbolev Space). *Given a measurable function $u : \Omega \to \mathbb{R}$ we define the following quantities,*

$$\|u\|_{\mathcal{W}^{m,2}(|\mathbf{x}|^{2\beta+2m},\Omega)} = \Big(\sum_{k=0}^{m} |u|^2_{\mathcal{W}^{k,2}(|\mathbf{x}|^{2\beta+2m},\Omega)}\Big)^{\frac{1}{2}},$$

$$|u|_{\mathcal{W}^{k,2}(|\mathbf{x}|^{2\beta+2m},\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|^2_{\mathcal{L}^2\left(|\mathbf{x}|^{2\beta+2m},\Omega\right)}\right)^{\frac{1}{2}}.$$

Furthermore we will call ***Maz'ya-Sobolev*** *space, $\mathcal{W}^{m,2}(|\mathbf{x}|^{2\beta+2m},\Omega)$, the set of measurable functions $u : \Omega \to \mathbb{R}$ with finite $\|u\|_{\mathcal{W}^{m,2}(|\mathbf{x}|^{2\beta+2m},\Omega)}$. Once again we will denote $\mathcal{W}_0^{m,2}(|\mathbf{x}|^{2\beta+2m},\Omega)$ the closure of the smooth functions with compact support with respect to the $\|\cdot\|_{\mathcal{W}^{m,2}(|\mathbf{x}|^{2\beta+2m},\Omega)}$.*

Theorem 1.5. *The triplet $\Big(H^1(\Omega), \mathcal{W}^{m,2}(|\mathbf{x}|^{2(\beta+m+2)},\Omega), \mathcal{W}^{m+2,2}(|\mathbf{x}|^{2(\beta+m+2)},\Omega)\Big)$ is a shift triplet for the weak formulation of* (2.4)*.*

Proof. I redirect the reader to interested in the proof of this result to [67], Chapter 7. □

Corollary 1.6. *The triplet $\Big(H^1(\Omega), \mathcal{W}^{m,2}(|\mathbf{x}|^{2(\beta+m+2)},\Omega), \mathcal{W}^{m+2,2}(|\mathbf{x}|^{2(\beta+m+2)},\Omega)\Big)$ is a shift triplet for* (1.18)*.*

Proof. The key to prove this result is the work by Z. Mghazli, [75], where it is showed using techniques similar to the one presented in [55] that (2.13) and the weak formulation of (2.4) present the same singular behaviour on polygonal domains. A careful reader might be bothered by the fact that Ω is not polygonal. Since we know that for smooth domains (1.18) enjoys a shift triplet like the one presented in Theorem 3.5, then we are only concerned by the singular part of the boundary which is of polygonal type. □

Before going to the next part of this chapter which will be focused on the approximation of functions in Munkenhoupt weighted Sobolev spaces we would like to address the existence of a Gagliardo-Nierenberg-Sobolev type embedding, i.e.

$$\mathcal{W}_0^{1,q}(|\mathbf{x}|^{2(\beta+2)},\Omega) \hookrightarrow \mathcal{L}^p((|\mathbf{x}|^{2(\beta+2)},\Omega). \tag{2.5}$$

In order to do this we will first discuss the existence of a general continuous embedding of the form,

$$\mathcal{W}_0^{1,q}(\omega,\Omega) \hookrightarrow \mathcal{L}^p(\rho,\Omega),$$

for Muckenhoupt weighted Sobolev space, then we will derive (2.5) as a particular case. I will first introduce a Lemma that is going to be fundamental to prove the following result.

Lemma 1.7. *Let F be a Lipschitz continuous function on $B_\tau(\mathbf{x_0}) \subset \mathbb{R}^2$, $p \in (1,q]$, $\rho \in A_q(\mathbb{R}^2)$ and $\omega \in A_p(\mathbb{R}^2)$ such that:*

$$\frac{r}{R}\left(\frac{\rho\big(B_r(\mathbf{x_0})\big)}{\rho\big(B_R(\mathbf{x})\big)}\right)^{\frac{1}{q}} \leq C_{\rho,\omega}\left(\frac{\omega\big(B_r(\mathbf{x_0})\big)}{\omega\big(B_R(\mathbf{x})\big)}\right)^{\frac{1}{p}}, \tag{2.6}$$

for all $x \in B_{2r}(1\mathbf{x_0})$ and $0 < r < R$. Then the following inequality holds,

$$\left(\frac{1}{\rho\big(B_R(\mathbf{x0})\big)}\int_{B_R(\mathbf{x})}|F(u)|^q\rho(u)\right)^{\frac{1}{q}} \leq Cr\left(\frac{1}{\omega\big(B_R(\mathbf{x0})\big)}\int_{B_R(\mathbf{x})}|\nabla F(u)|^p\omega(u)\right)^{\frac{1}{p}}.$$

Proof. The proof of this result can be found for $p \in (1,q)$ in [24]. Furthermore it has been proven for Holder continuous function when $p = q$ in [73]. □

Theorem 1.8. *Given two weights ω,ρ that satisfy the hypothesis of the previous Lemma then there exists a continuous embedding,*

$$\mathcal{W}_0^{1,q}(\omega,\Omega) \hookrightarrow \mathcal{L}^p(\rho,\Omega).$$

Proof. In order to prove this result we first embed Ω in a ball of radius R and consider $\widetilde{v}$ the extension of v by zero outside of Ω. We first notice that when $p = q$ thanks to Morrey's inequality we can apply the previous Lemma and conclude. When $p \in (1,q)$ we need to argue by density i.e. we consider a sequence $v_n \in C_c^\infty(\Omega)$ such that $v_n \overset{\mathcal{W}_0^{1,2}(\omega,\Omega)}{\longrightarrow} \widetilde{v}$ and observe that for all $n \in \mathbb{N}$ the following inequalities follow from the previous Lemma,

$$\|v_n\|_{\mathcal{L}^q\left(\rho,B_R(\mathbf{x_0})\right)} \leq R\rho\big(B_R(\mathbf{x_0})\big)^{\frac{1}{q}}\omega\big(B_R(\mathbf{x_0})\big)^{-\frac{1}{p}}\|\nabla w\|_{\mathcal{L}^p\left(\omega,B_R(\mathbf{x_0})\right)}.$$

Bringing the above expression to the limit and observing that $\rho\omega$ satisfies the strong doubling property we have,

$$\|v_n\|_{\mathcal{L}^q\left(\rho,\Omega\right)} \leq C(C_{q,\rho},C_{p,\omega})diam(\Omega)\rho\big(\Omega\big)^{\frac{1}{q}}\omega\big(\Omega\big)^{-\frac{1}{p}}\|\nabla w\|_{\mathcal{L}^p\left(\omega,\Omega\right)},$$

which gives us the desired embedding. □

Remark 1.9. I urge the reader to notice that we are developing an embedding of $\mathcal{W}_0^{1,q}(\omega,\Omega)$ in $\mathcal{L}^p(\rho,\Omega)$ and not of $\mathcal{W}^{1,q}(\omega,\Omega)$. From the proof of the above Theorem the reason appears clear, in fact given the fact we work with domains that do not have smooth boundaries it would not be possible to extend u by zero outside of Ω.

Corollary 1.10. *If $p \in (1,q]$, $\gamma \in \big(-2,2(p-1)\big)$ and*

$$\frac{2(\beta+3)+1}{2(\beta+3)} \leq \frac{q}{p},$$

then the continuous embedding $\mathcal{W}_0^{1,q}(|\mathbf{x}|^{2(\beta+2)}, \Omega) \hookrightarrow \mathcal{L}^p(|\mathbf{x}|^{2(\beta+2)}, \Omega)$ *exists.*

Proof. To obtain this result it is just a matter of combining the above result with Proposition 1.1 and computing exactly all the terms involved. □

2 Approximation of Functions in Weighted Sobolev Spaces

We are interested in approximating the functions of $\mathcal{W}^{2,2}\Big(\Omega, |x|^{2(\beta+1)}\Big)$. In order to do this I will introduce the quasi interpolant developed in [79], for the piecewise linear case. Other examples of weighted quasi interpolant can be found in [43] and [34].

Definition 2.1. *A domain $\Omega \subset \mathbb{R}^2$ is a star-shaped with respect to a ball $B \subset \Omega$, if for all $x \in \Omega$ there exists $X_B \in B$ such that the line connecting x_B and x is entirely contained in Ω.*

The first result we need in order to construct the interpolant is a Poincaré type inequality for weighted Sobolev spaces, similar to the one in Lemma 2.10.

Lemma 2.2. *Let $\omega \in A_p(\mathbb{R}^2)$, with $p \in (1, \infty)$, Ω pre-compact in $\mathbb{R}^2$ that verifies the above definition, then given $f \in \mathcal{L}_0^p(\omega, \Omega)$ there exists $\mathbf{u} \in \big[W_0^{1,p}(\omega, \Omega)\big]^n$ such that, $\nabla \cdot \mathbf{u} = f$, and the following inequality holds,*

$$\|u\|_{\big[W^{1,p}(\omega,\Omega)\big]^n} \leq C\|f\|_{\mathcal{L}^p(\omega,\Omega)}.$$

Proof. This result in the non weighted setting is a consequence of the Banach close range theorem and Ladyzhenskaya Theorem on surjectivity of the divergence. The same result in the context of weighted Sobolev space has been proven by R. G. Duran and F. L. Garcia in [40]. □

Theorem 2.3 (Weighted Poincaré Inequality). *Let $\Omega \subset \mathbb{R}^2$ be as in the above definition, with $diam(\Omega) \approx 1$. Furthermore let $\chi \in C^0(\Omega)$, such that $\int_\Omega \chi\, d\mathbf{x} = 1$. Then fixed $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R}^2)$, for all $v \in \mathcal{W}^{1,p}(\mu, \Omega)$ such that $\int_\Omega v\chi\, d\mathbf{x} = 0$ we have:*

$$\|v\|_{\mathcal{L}^p(\mu,\Omega)} \leq C\|\nabla v\|_{\mathcal{L}^p(\mu,\Omega)},$$

where μ is an isotropic dilation - translation of ω and C depends only on $\chi, B, C_{p,\omega}$.

Proof. We know from Theorem 0.3 that $\mu \in A_p(\mathbb{R}^2)$ and $C_{p,\omega} = C_{p,\mu}$. Given $v \in \mathcal{W}^{1,p}(\mu, \Omega)$ we define the auxiliary function,

$$\widetilde{v} = sign(v)|v|^{p-1}\mu - \Big(\int_\Omega sign(v)|v|^{p-1}\mu\Big)\chi\, d\mathbf{x}.$$

If we apply Holder inequality to the previous equation we get,

$$\int_\Omega \mu|v|^{p-1}\mu^{\frac{1}{p}}\mu^{\frac{1}{q}}\,d\mathbf{x} \le \Big(\int_\Omega \mu|v|^p\,d\mathbf{x}\Big)^{\frac{1}{q}}\Big(\int_\Omega \mu\,d\mathbf{x}\Big)^{\frac{1}{p}} \le C(\mu,p)\|v\|^{\frac{p}{q}}_{\mathcal{L}^p(\mu,\Omega)} \le C(\mu,p)\|v\|^{p-1}_{\mathcal{L}^p(\mu,\Omega)}. \tag{2.7}$$

We define $t = -\frac{p'}{p}$ then $t+q=1$ and $q(p-1)=p$ therefore,

$$\Big(\int_\Omega \mu^t|v|^q\,d\mathbf{x}\Big)^{\frac{1}{q}} = \left(\int_\Omega \mu^t\left|sign(v)|v|^{p-1}\mu - \Big(\int_\Omega sign(v)|v|^{p-1}\mu\Big)\chi\right|^q d\mathbf{x}\right)^{\frac{1}{q}}$$
$$\le \Big(\int_\Omega \mu^{t+q}|v|^{q(p-1)}\,d\mathbf{x}\Big)^{\frac{1}{q}} + \Big(\int_\Omega |v|^{p-1}\mu\,d\mathbf{x}\Big)\|\chi\|_{L^q(\mu^t,\Omega)} \overset{(2.7)}{\le} \|v\|^{p-1}_{\mathcal{L}^p}(\mu,\Omega)$$

Once again we use Theorem 0.3 to assert that $\mu^t \in A_q(\mathbb{R}^2)$ since $t=\frac{q}{p}$, in this way we can apply the previous Lemma since $\int_\Omega \chi = 1$ implies $\displaystyle\int_\Omega \widetilde{v}\,d\mathbf{x} = 0$, i.e. there exists $\mathbf{u} \in \big[W^{1,q}_0(\mu^t,\Omega)\big]^n$ such that,

$$\|u\|_{\left[W^{1,q}(\mu^t,\Omega)\right]^n} \le C\left\|\widetilde{v}\right\|_{\mathcal{L}^q(\mu^t,\Omega)}. \tag{2.8}$$

Finally we notice that since by hypothesis $\displaystyle\int_\Omega v\chi\,d\mathbf{x} = 0$, then we have,

$$\|v\|^p_{\mathcal{L}^p(\mu,\Omega)} = \int_\Omega v\widetilde{v}\,d\mathbf{x} + \Big(\int_\Omega sign(v)|v|^{p-1}\mu\,d\mathbf{x}\Big)\int_\Omega \chi v\,d\mathbf{x} = \int_\Omega v\widetilde{v}\,d\mathbf{x}.$$

Substituting $\widetilde{v}$ by $\nabla\cdot u$ we get,

$$\|v\|^p_{\mathcal{L}^p(\mu,\Omega)} = \int_\Omega v(\nabla\cdot u)\,d\mathbf{x} \le \int_\Omega |u\nabla v|\,d\mathbf{x} = \int_\Omega \left|u\nabla v\mu^{\frac{1}{p}}\mu^{-\frac{1}{q}}\right|d\mathbf{x}$$
$$\overset{H.}{\le} \Big(\int \mu|\nabla v|^p\,d\mathbf{x}\Big)^{\frac{1}{p}}\Big(\int \mu^t|u|^q\,d\mathbf{x}\Big)^{\frac{1}{q}} \overset{(2.8)}{\le} \|\nabla v\|_{\mathcal{L}^p(\mu,\Omega)}\left\|\widetilde{v}\right\|_{\mathcal{L}^p(\mu,\Omega)} \overset{(2.7)}{\le} \|\nabla v\|_{\mathcal{L}^p(\mu,\Omega)}\|v\|^{p-1}_{\mathcal{L}^p(\mu,\Omega)}.$$

□

Since we have proven a weighted Poincare inequality we can begin the construction of a piecewise linear quasi interpolant for functions in $\mathcal{W}^{m,p}(\omega,\Omega)$. To do so I will begin by some geometric assumption on the mesh.

Definition 2.4. *We will call $\mathcal{T} = \Big\{T_i : i\in I\}$ a **simplicial mesh** of a d dimensional polytope Ω, if T_i are simplexes, $\overline{\Omega} = \cup_{i\in I}T_i$ and $|\Omega| = \sum_{i\in I}|T_i|$. Furthermore we say that the mesh is **compatible** if the intersection of two $T_i, T_j \in \mathcal{T}$ is either empty of a common $d-1$ dimensional simplex. Last we say that the mesh $\mathcal{T}$ is **shape regular** if there exists $\sigma_{\mathcal{T}} > 0$ such that,*

$$\max\left\{\frac{h_T}{\rho_T} : T\in\mathcal{T}\right\} \le \sigma_{\mathcal{T}}, \tag{2.9}$$

where h_T is the diameter of T and ρ_T is the diameter of the inscribed sphere in T.

I will focus my attention on the two dimensional case, mainly. Given a simplex $T \in \mathcal{T}$ we denote $\mathcal{N}(T)$ the vertices of the simplex T and $\overset{\circ}{\mathcal{N}}(T)$ the vertices of the simplex T that do not touch the boundary of Ω. Furthermore the union of $\mathcal{N}(T)$ for all $T \in \mathcal{T}$ I will denote by $\mathcal{N}(\mathcal{T})$, similarly the same notation $\overset{\circ}{\mathcal{N}}(\mathcal{T})$ is adopted to represent to union $\overset{\circ}{\mathcal{N}}(T)$ for all $T \in \mathcal{T}$. As usually done in the finite element method I introduce the spaces V and $\overset{\circ}{V}$ as follows,

$$V = \Big\{ v \in C^0(\overline{\Omega})\,;\, w_T \in \mathbb{P}_1(T)\,\forall T \in \mathcal{T} \Big\},$$
$$\overset{\circ}{V} = \Big\{ v \in C^0(\overline{\Omega})\,;\, w_T \in \mathbb{P}_1(T)\,\forall T \in \mathcal{T} \text{ and } \gamma_0(v) \equiv 0 \Big\}. \tag{2.10}$$

In particular any function of $v \in V$ or $v_0 \in \overset{\circ}{V}$ can be uniquely determined by its degrees of freedom, i.e.

$$v = \sum_{\mathbf{z} \in N(\mathcal{T})} v_{\mathbf{z}} \phi_{\mathbf{z}}, \qquad v_0 = \sum_{\mathbf{z} \in \overset{\circ}{N}(\mathcal{T})} v_{\mathbf{z}} \phi_{\mathbf{z}}, \tag{2.11}$$

where $v_{\mathbf{z}}$ is the degree of freedom associated to the base function $\phi_{\mathbf{z}}$, which is the value of v at the node $\mathbf{z}$. Last we will call $S_{\mathbf{z}}$ the union of the elements that contains $\mathbf{z}$ inside and S_T the union of all elements of the mesh that have non empty intersection with T. We now consider $\psi \in C^\infty(\mathbb{R}^n)$ such that $\|\psi\|_{\mathcal{L}^1(\mathbb{R}^2)} = 1$ such that the support of ψ is contained in $B_{R(\sigma_{\mathcal{T}})}(\mathbf{0})$, which will be our averaging function. Since we need to average inside each element of the triangulation we define the following scaling of ψ,

$$\psi_{\mathbf{z}}(\mathbf{x}) = \frac{(m+1)^2}{h_{\mathbf{z}}} \psi\Big(\frac{(m+1)(\mathbf{z}-\mathbf{x})}{h_{\mathbf{z}}}\Big).$$

Now for all $\mathbf{z} \in \overset{\circ}{\mathcal{N}}$ and $v \in \mathcal{W}^{m,p}(\omega, \Omega)$ we define the **averaged Taylor polynomial**, also known as Sobolev polynomial, of order m around $\mathbf{z}$, as:

$$Q^m_{\mathbf{z}} v(\mathbf{y}) = \int_{S_{\mathbf{z}}} P^m v(\mathbf{x}, \mathbf{y}) \psi_{\mathbf{z}}(\mathbf{x})\, d\mathbf{x},$$

where $P^m v(\mathbf{x}, \mathbf{y})$ is the usual Taylor polynomial, i.e. $P^m v(\mathbf{x}, \mathbf{y}) = \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha v(\mathbf{x})(\mathbf{y}-\mathbf{x})^\alpha$.

Remark 2.5. The reader now can clearly see why it is important that the support of $\psi_{\mathbf{z}}$ is contained in $S_{\mathbf{z}}$. In fact if $supp(\psi_{\mathbf{z}}) \subset S_{\mathbf{z}}$ then we can integrate by parts to show that $Q^m_{\mathbf{z}} v$ is well defined for all $v \in \mathcal{L}^1(\Omega)$. Furthermore since Ω is pre-compact and $L^p(\omega, \Omega) \subset \mathcal{L}^1_{loc}(\Omega)$ then $Q^m_{\mathbf{z}} v$ is well defined for all $v \in \mathcal{L}^p(\omega, \Omega)$.

It is now time to discuss the property of averaged Taylor polynomial just introduced; the following proposition will be a generalisation for weighted Sobolev space of the concept introduced in [20], Chapter 4.

Proposition 2.6. *Given a function $v \in \mathcal{L}^p(\omega, \Omega)$ the object $Q^m_{\mathbf{z}} v$, as defined above, enjoys the following properties:*

1. *$Q^m_{\mathbf{z}}v$ is a polynomial of degree at most m,*
2. *$Q^m_{\mathbf{z}}v$ is a projection,*
3. *$D^\alpha Q^m_{\mathbf{z}}v = Q^{m-|\alpha|}D^\alpha v$, for all $v \in \mathcal{W}^{|\alpha|,p}(\omega,\Omega)$ and $\alpha \in \mathbb{N}^2$ such that $|\alpha| \le 1$.*

Proof. 1. Since we are integrating in the variable $\mathbf{x}$ and $P^m v(\mathbf{x},\mathbf{y})$ is a polynomial in $\mathbf{y}$ the first property comes for free.

2. The key ingredient to prove this is that the Taylor polynomial is a projection.

$$\begin{aligned}
Q^m_{\mathbf{z}}\left[Q^m_{\mathbf{z}}v\right] &= \int_{S_{\mathbf{z}}} P^m\Big[\int_{S_{\mathbf{z}}} P^m v(\mathbf{x},\mathbf{y})\psi_{\mathbf{z}}(\mathbf{x})\,d\mathbf{x}\Big](\mathbf{y},\mathbf{z})\psi(\mathbf{y})\,d\mathbf{y}\\
&= \int_{S_{\mathbf{z}}}\int_{S_{\mathbf{z}}} P^m P^m v(\mathbf{x},\mathbf{y})(\mathbf{y},\mathbf{z})\psi_{\mathbf{z}}(\mathbf{x})\psi_{\mathbf{z}}(\mathbf{y})\,d\mathbf{x}d\mathbf{y}\\
&=\left(\int_{S_{\mathbf{z}}} P^m v(\mathbf{y},\mathbf{z})\psi(\mathbf{y})\,d\mathbf{y}\right)\left(\int_{S_{\mathbf{z}}}\psi(\mathbf{x})\,d\mathbf{x}\right) = Q^m_{\mathbf{z}}v.
\end{aligned}$$

3. First we notice that if $v \in W^{|\alpha|,1}(\Omega)$, then $D^\alpha v \in \mathcal{L}^1(\Omega)$ and therefore it make sense to speak about , $Q^{m-|\alpha|}_{\mathbf{z}}D^\alpha v$. Now we consider $v_n \in C^\infty(\Omega)$ such that $v_n \overset{\mathcal{W}^{|\alpha|,1}(\Omega)}{\longrightarrow} v$, and notice that:

$$D^\alpha Q^m_{\mathbf{z}}v_n(\mathbf{x}) = \int_{S_{\mathbf{z}}} D^\alpha P^m v_n(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y} = \int_{S_{\mathbf{z}}} P^{m-|\alpha|}v_n(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y} = Q^{m-|\alpha|}D^\alpha v_n(\mathbf{x}).$$

Passing the above expression to the limit we obtain the wanted equality for $v \in \mathcal{W}^{1,1}(\Omega)$. To conclude we observe that following the same argument as in Remark 2.5, we have $\mathcal{W}^{1,1}(\omega,\Omega) \subset \mathcal{W}^{1,1}(\Omega)$.

□

Lemma 2.7. *Let $\omega \in A_p(\mathbb{R}^2)$ and $\mathbf{z} \in \overset{\circ}{\mathcal{N}}(\mathcal{T})$, if $v \in \mathcal{W}^{k,p}(\omega,S_{\mathbf{z}})$ with $0 \le k \le 1$ the following inequality holds,*

$$\left\|Q^1_{\mathbf{z}}\right\|_{L^\infty(S_{\mathbf{z}})} \le C(m,\psi)h_{\mathbf{z}}^{-2}\|1\|_{\mathcal{L}^q(\omega^{-\frac{q}{p}})}\sum_{l=0}^{k} h^l_{\mathbf{z}}|v|_{\mathcal{W}^{l,p}(\omega,S_{\mathbf{z}})},$$

where $h_{\mathbf{z}}$ is the largest diameter of the element that form $S_{\mathbf{z}}$.

Proof. From the definition of $Q^m_{\mathbf{z}}$ we have,

$$\begin{aligned}\|Q_{\mathbf{z}}^m\|_{L^\infty(S_{\mathbf{z}})} =&\left\|\int_{S_{\mathbf{z}}}\sum_{|\alpha|\le m}\frac{1}{\alpha!}D^\alpha v(\mathbf{x})(\mathbf{y}-\mathbf{x})^\alpha\psi_{\mathbf{z}}(\mathbf{x})\,d\mathbf{x}\right\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}\\ \le&\frac{1}{\alpha!}\sum_{|\alpha|\le m}\left\|\int_{S_{\mathbf{z}}}D^\alpha v(\mathbf{x})(\mathbf{y}-\mathbf{x})^\alpha\psi_{\mathbf{z}}(\mathbf{x})\,d\mathbf{x}\right\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}\\ \le&\frac{1}{\alpha!}\sum_{|\alpha|\le m}\left\|\int_{S_{\mathbf{z}}}D^\alpha v(\mathbf{x})(\mathbf{y}-\mathbf{x})^\alpha\psi_{\mathbf{z}}(\mathbf{x})\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}\,d\mathbf{x}\right\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}.\end{aligned}$$

If we fix $k\in[0,m]$ from the above inequality we get,

$$\begin{aligned}\|Q_{\mathbf{z}}^m\|_{L^\infty(S_{\mathbf{z}})}\le&\frac{1}{\alpha!}\sum_{|\alpha|\le m}\left\|\int_{S_{\mathbf{z}}}D^\alpha v(\mathbf{x})(\mathbf{y}-\mathbf{x})^\alpha\psi_{\mathbf{z}}(\mathbf{x})\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}\,d\mathbf{x}\right\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}\\ \le&\frac{1}{\alpha!}\sum_{|\alpha|\le m}\left\|\int_{S_{\mathbf{z}}}D^k v(\mathbf{x})D^{\alpha-k}\big[(\mathbf{y}-\mathbf{x})^\alpha\psi_{\mathbf{z}}(\mathbf{x})\big]\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}\,d\mathbf{x}\right\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}\\ \le&\frac{1}{\alpha!}\sum_{|\alpha|\le m}\left\|\int_{S_{\mathbf{z}}}D^k v(\mathbf{x})D^{\alpha-k}\big[h_{\mathbf{z}}^\alpha\psi_{\mathbf{z}}(\mathbf{x})\big]\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}\,d\mathbf{x}\right\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}\\ \le&\frac{1}{\alpha!}\sum_{|\alpha|\le m}h_{\mathbf{z}}^\alpha h_{\mathbf{z}}^{-2}\big\|D^{\alpha-k}\psi(\mathbf{x})\big\|_{\mathcal{L}^\infty(S_{\mathbf{z}})}\|1\|_{\mathcal{L}^q(\omega^{-\frac{q}{p}})}\big\|D^k v(\mathbf{x})\big\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}\\ \le&C(m,\psi)\|1\|_{\mathcal{L}^q(\omega^{-\frac{q}{p}})}h_{\mathbf{z}}^{-2}\sum_{l=0}^m h_{\mathbf{z}}^l\big\|D^k v(\mathbf{x})\big\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}.\end{aligned}$$

□

Let us start discussing the approximation property of the averaged Taylor polynomial $Q_{\mathbf{z}}^0 v$ and $Q_{\mathbf{z}}^1 v$, for functions $v\in\mathcal{W}^{1,p}(\omega,S_{\mathbf{z}})$.

Lemma 2.8. *Let $z\in\overset{\circ}{\mathcal{N}}(\mathcal{T})$. If $v\in\mathcal{W}^{1,p}(\omega,S_{\mathbf{z}})$ then the following approximation estimate holds,*

$$\big\|v-Q_{\mathbf{z}}^0\big\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}\le Ch_{\mathbf{z}}\|\nabla v\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}.$$

Furthermore the following inequality holds,

$$\big\|D_{x_j}(v-Q_{\mathbf{z}}^1)\big\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}\le Ch_{\mathbf{z}}\big\|D_{x_j}\nabla v\big\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}.$$

Proof. We define the transformation,

$$F_{\mathbf{z}}:\mathbf{x}\mapsto\overline{\mathbf{x}},\qquad \overline{\mathbf{x}}=\frac{\mathbf{z}-\mathbf{x}}{h_{\mathbf{z}}},$$

we will also define $\overline{S}_{\mathbf{z}}=F_{\mathbf{z}}(S_{\mathbf{z}})$ and $\overline{v}(\overline{x})=v(x)$. Let $\overline{Q}^0\overline{v}$ be defined as follows,

$$\overline{Q}^0\overline{v}=\int_{S_{\mathbf{z}}}\overline{v}\psi\,d\overline{\mathbf{x}},$$

then thanks to the way we defined F_z we have that the support of ψ is contained in $S_{\mathbf{z}}$.

$$Q^0_{\mathbf{z}} = \int_{S_{\mathbf{z}}} v\psi_{\mathbf{z}}\, d\mathbf{x} = \int_{\overline{S}_{\mathbf{z}}} \overline{v}\psi_{\mathbf{z}}(\overline{\mathbf{x}})h^2_{\mathbf{z}}\, d\overline{\mathbf{x}} = \int_{\overline{S}_{\mathbf{z}}} \overline{v}\psi(\overline{\mathbf{x}})\, d\overline{\mathbf{x}} = \overline{Q}^0\overline{v}.$$

Using Theorem 0.3 we have that $\overline{\omega}_{\mathbf{z}} = \omega \circ F^{-1}_{\mathbf{z}} \in A_p(\mathbb{R}^2)$, since $F_{\mathbf{z}}$ is a translation combined with an isotropic dilation. Combining the last two statement together with the fact $C_{p,\overline{\omega}_{\mathbf{z}}} = C_{p,\omega_{\mathbf{z}}}$ we have the following estimate,

$$\int_{S_{\mathbf{z}}} \omega\left|v - Q^0_{\mathbf{z}}v\right|^p d\mathbf{x} = h^2_{\mathbf{z}} \int_{S_{\mathbf{z}}} \overline{\omega}\left|\overline{v} - Q^0_{\mathbf{z}}\overline{v}\right|^p d\overline{\mathbf{x}}. \tag{2.12}$$

Given the assumption (2.9) we have $diam(S_{\mathbf{z}}) \approx 1$, combined with $\int_{\overline{S}_{\mathbf{z}}} \psi\, d\overline{\mathbf{x}} = 1$ we have that $\int_{\overline{S}_{\mathbf{z}}} \overline{v} - \overline{Q}^0\overline{v} = 0$, which will allows us to apply Theorem 2.3 to conclude,

$$\left\|\overline{v} - \overline{Q}^0_{\mathbf{z}}\overline{v}\right\|_{\mathcal{L}^p(\overline{\omega}_{\mathbf{z}},\overline{S}_{\mathbf{z}})} \leq C(\sigma_{\mathcal{T}}, \overline{\omega}_{\mathbf{z}}, \psi)\|\nabla_{\overline{x}}\overline{v}\|_{\mathcal{L}^p(\overline{\omega}_{\mathbf{z}},\overline{S}_{\mathbf{z}})}.$$

Combining this last inequality with (2.12) and changing variable from $\overline{\mathbf{x}}$ to $\mathbf{x}$ we have,

$$\begin{aligned}\left\|v - Q^0_{\mathbf{z}}v\right\|_{\mathcal{L}^p(\omega_{\mathbf{z}},S_{\mathbf{z}})} =& h^2_{\mathbf{z}} \int_{S_{\mathbf{z}}} \overline{\omega}\left|\overline{v} - Q^0_{\mathbf{z}}\overline{v}\right|^p d\overline{\mathbf{x}} \\ \leq& h^2_{\mathbf{z}} C(\sigma_{\mathcal{T}}, \overline{\omega}_{\mathbf{z}}, \psi)\|\nabla_{\overline{x}}\overline{v}\|_{\mathcal{L}^p(\overline{\omega}_{\mathbf{z}},\overline{S}_{\mathbf{z}})} = h^2_{\mathbf{z}} C(\sigma_{\mathcal{T}}, \overline{\omega}_{\mathbf{z}}, \psi)\|\nabla_x v\|_{\mathcal{L}^p(\omega_{\mathbf{z}},S_{\mathbf{z}})}.\end{aligned}$$

In order to estimate $\left\|D_{x_j}(v - Q^1_{\mathbf{z}})\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}$ we define $\overline{Q}^1\overline{v}(\overline{y})$ as follows,

$$\overline{Q}^1\overline{v}(\overline{y}) = \int_{\overline{S}_{\mathbf{z}}} \Big(\overline{v}(\overline{x}) + \nabla_{\overline{x}}\overline{v}(\overline{x}) \cdot (\overline{y} - \overline{x})\Big)\psi(\overline{x})\, d\overline{x}.$$

As in the previous case we have the identity $Q^1_{\mathbf{z}}(\mathbf{y}) = \overline{Q}^1\overline{v}(\overline{y})$, furthermore since $\overline{Q}^1\overline{v}(\overline{y})$ the quantity $\partial_{\overline{y}_i}\overline{Q}^1\overline{v}(\overline{y})$ is constant and therefore,

$$\int_{\overline{S}_{\mathbf{z}}} \partial_{\overline{x}_i}\Big(\overline{v}(\overline{x}) - \overline{Q}^1\overline{v}(\overline{x})\Big)\psi(\overline{x})\, d\overline{x} = 0.$$

Since $\partial_{\overline{x}_i}\Big(\overline{v}(\overline{x}) - \overline{Q}^1\overline{v}(\overline{x})\Big)$ has vanishing mean we can follow the same argument presented above to obtain,

$$\left\|D_{x_j}(v - Q^1_{\mathbf{z}})\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})} \leq Ch_{\mathbf{z}}\left\|D_{x_j}\nabla v\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})}.$$

□

Lemma 2.9. *Let $z \in \mathring{\mathcal{N}}$ and $v \in \mathcal{W}^{2,p}(\omega, S_{\mathbf{z}})$ then we have the following approximation estimate,*

$$\left\|v - Q^1_{\mathbf{z}}\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})} \le C\big(C_{p,\omega},\sigma_{\mathcal{T}},\psi\big)h^2_{\mathbf{z}}\|v\|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{z}})}.$$

Proof. First we apply Proposition 2.6 to observe the following,

$$v - Q^1_{\mathbf{z}}v = \big(v - Q^1_{\mathbf{z}}v\big) - Q^0_{\mathbf{z}}\Big(v - Q^1_{\mathbf{z}}v\Big) - Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}v - v\Big), \qquad \nabla\big(v - Q^1_{\mathbf{z}}v\big) = \nabla v - Q^0_{\mathbf{z}}\nabla v. \quad (2.13)$$

If we apply the previous Lemma twice we then obtain,

$$\left\|\big(v - Q^1_{\mathbf{z}}v\big) - Q^0_{\mathbf{z}}\Big(v - Q^1_{\mathbf{z}}v\Big)\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})} \le Ch_{\mathbf{z}}\left\|\nabla\Big(v - Q^1_{\mathbf{z}}\Big)\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})} \le Ch^2_{\mathbf{z}}|v|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{z}})}.$$

Therefore from (2.13) the only term we need to control is $Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}v - v\Big)$, which using again Proposition 2.6 is equivalent to controlling $Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}v - Q^0_{\mathbf{z}}v\Big)$. Now we notice that for all linear polynomials p we have $Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}p - Q^0_{\mathbf{z}}p\Big) = 0$ and therefore we can add and subtract $Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}Q^1_{\mathbf{z}}v - Q^0_{\mathbf{z}}Q^1_{\mathbf{z}}\Big)$ freely. We start from $Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}v - Q^0_{\mathbf{z}}v\Big)$ and we subtract $Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}Q^1_{\mathbf{z}}v - Q^0_{\mathbf{z}}Q^1_{\mathbf{z}}\Big)$ to obtain,

$$\begin{aligned}
\left\|Q^0_{\mathbf{z}}\Big(Q^1_{\mathbf{z}}v - v\Big)\right\|_{\mathcal{L}^p(\omega,S_{\mathbf{z}})} &= \int_{S_{\mathbf{z}}}\omega\int_{S_{\mathbf{z}}}\left|\nabla\big(v(x) - Q^1_{\mathbf{z}}\cdot(\mathbf{y}-\mathbf{x})\psi_{\mathbf{z}}\big)\,d\mathbf{y}\right|^p \\
\overset{H.}{\le} \int_{S_{\mathbf{z}}}\omega\left|h^p_{\mathbf{z}}\Big(\int_{S_{\mathbf{z}}}\omega\left|\nabla(v(x) - Q^1_{\mathbf{z}}v(x))\right|^p d\mathbf{x}\Big)\Big(\int_{S_{\mathbf{z}}}\omega^{-\frac{q}{p}}\psi^q_{\mathbf{z}}\Big)^{\frac{p}{q}}\right| &\le C\big(C_{p,\omega},\sigma_{\mathcal{T}},\psi\big)h^{2p}_{\mathbf{z}}|v|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{z}})},
\end{aligned}$$

where in order to obtain the last inequality we used the fact that $\int_{S_{\mathbf{z}}}\psi_{\mathbf{z}}(y)\,dy = 1$ and $\Big(\int_{S_{\mathbf{z}}}\omega^{-\frac{q}{p}}\psi^q_{\mathbf{z}}\Big)$ is bounded. □

I don't need to develop this argument further in order to obtain an approximation estimate also for $v - Q^m_{\mathbf{z}}$. This because in later chapters I will only work with linear finite element schemes, but I redirect the reader interested in this results to [79]. Keeping in mind the notation used in (2.11) it is time to introduce an interpolant for function that are in $\mathcal{W}^{2,p}(\omega,\Omega)$. In particular given a function $v \in \mathcal{W}^{2,p}(\omega,\Omega)$ we define the interpolant $I^1_{\mathcal{N}}$ as follows,

$$I^1_{\mathcal{N}}v = \sum_{z\in\mathring{\mathcal{N}}(\mathcal{T})} Q^1_z v(z)\phi_z. \quad (2.14)$$

Now we will use Lemma 2.7 in order to prove the stability of the interpolant operator $I^1_{\mathcal{N}}$ with respect to the Muckenhoupt weighted Sobolev space $\mathcal{W}^{2,p}(\omega,S_{\mathbf{T}})$.

Proposition 2.10. *Let $\mathcal{W}^{2,p}(\omega,S_{\mathbf{z}})$ and T be an element of the triangulation $\mathcal{T}$, then the interpolant operator $I^1_{\mathcal{N}}$ is stable with respect to the Muckenhoupt weighted Sobolev space $\mathcal{W}^{2,p}(\omega,S_{\mathbf{T}})$, i.e.*

$$\left|I^1_{\mathcal{N}}v\right|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{T}})} \le |v|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{T}})}.$$

Proof. Using the definition of $I^1_{\mathcal{N}}$, given in (2.14) we have that,

$$|I_{\mathcal{N}}^1 v|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{T}})} \leq \sum_{z \in \overset{\circ}{\mathcal{N}}(\mathcal{T})} \|Q_z^1(z)\|_{\mathcal{L}^\infty(\mathcal{S}_{\mathbf{z}})} |\phi_z|_{\mathcal{W}^{2,p}(\omega,S_{\mathbf{T}})}.$$

Combining the last inequality with Lemma 2.7 we can conclude. □

Lemma 2.11. *Given a function $\varphi \in \mathcal{L}^1(\omega, S_{\mathbf{z}})$ the following identity holds,*

$$I_{\mathcal{N}}^1 Q_{\mathbf{z}}^1 \varphi = Q_{\mathbf{z}}^1 \varphi.$$

Proof. This is a consequence of the fact that $Q_{\mathbf{z}}^1 p = p$ for all linear polynomials p and of (2.14). □

Theorem 2.12. *Given $T \in \mathcal{T}$ such that T is not a simplex with vertex on the boundary of Ω and $v \in \mathcal{W}^{2,p}(\omega, S_T)$ we then have the following interpolation estimate,*

$$|v - I_{\mathcal{N}}^1 v|_{\mathcal{W}^{k,p}(\omega,T)} \leq C h_T^{2-k} |v|_{\mathcal{W}^{2,p}(\omega,S_T)}.$$

Proof. Since the simplex T doesn't touch the boundary we consider one vertex $\mathbf{z}$ of T such that $\mathbf{z} \in \overset{\circ}{\mathcal{N}}$. From the previous lemma we know that the following identity holds,

$$v - I_{\mathcal{N}}^1 = v - Q_{\mathbf{z}}^1 v + I_{\mathcal{N}}^1 Q_{\mathbf{z}}^1 v - I_{\mathcal{N}}^1 v,$$

and therefore a simple application of the triangular inequality yields:

$$|v - I_{\mathcal{N}}^1|_{\mathcal{W}^{k,p}(\omega,T)} \leq |v - Q_{\mathbf{z}}^1 v|_{\mathcal{W}^{k,p}(\omega,S_T)} + |I_{\mathcal{N}}^1 Q_{\mathbf{z}}^1 v - I_{\mathcal{N}}^1 v|_{\mathcal{W}^{k,p}(\omega,S_T)}.$$

The last inequality together with Proposition 2.10, Lemma 2.8 and Lemma 2.9, yields the desired interpolation estimate. □

Corollary 2.13. *Given $T \in \mathcal{T}$ and $v \in \mathcal{W}^{2,p}(\omega, S_T) \cap \mathcal{W}_0^{1,p}(\omega, S_T)$ we then have the following interpolation estimate,*

$$|v - I_{\mathcal{N}}^1 v|_{\mathcal{W}^{k,p}(\omega,T)} \leq C h_T^{2-k} |v|_{\mathcal{W}^{2,p}(\omega,S_T)}.$$

Corollary 2.14. *Given $\mathcal{T}$ as in* (2.1)*, that also verifies* (2.9)*, and $v \in \mathcal{W}^{2,p}(\omega, \Omega) \cap \mathcal{W}_0^{1,p}(\omega, \Omega)$ we then have the following interpolation estimate,*

$$|v - I_{\mathcal{N}}^1 v|_{\mathcal{W}^{k,p}(\omega,\Omega)} \leq C h_T^{2-k} |v|_{\mathcal{W}^{2,p}(\omega,\Omega)}.$$

It is also possible to prove interpolation result when using norms of different Muckenhoupt weighted Sobolev spaces using embedding (2.5).

Proposition 2.15. *Let $p \in (1, q]$, $\rho \in A_q(\mathbb{R}^2)$, $\omega \in A_p(\mathbb{R}^2)$ such that* (2.6) *holds. If $v \in \mathcal{W}^{1,p}(\omega, \Omega)$ then there exists a constant v_Ω such that the following inequality holds,*

$$\|v - v_\Omega\|_{\mathcal{L}^q(\rho,\Omega)} \leq C\, diam(\Omega) \rho(\Omega)^{\frac{1}{q}} \omega(\Omega)^{-\frac{1}{p}} \|\nabla v\|_{\mathcal{L}^p(\omega,\Omega)}.$$

Proof. Since Ω is open and bounded we can choose $0 < r < R$ such that the following chain of inclusions hold,

$$\overline{B}_r(\mathbf{0}) \subset \Omega \subset \overline{\Omega} \subset B_R(\mathbf{0}).$$

We can use the extension theorem for Muckenhoupt weighted Sobolev spaces proven in [26], to find a $\widetilde{v} \in \mathcal{W}^{1,p}\big(\omega, B_R(\mathbf{0})\big)$ such that,

$$\left\|\nabla\widetilde{v}\right\|_{\mathcal{L}^p\left(\omega,B_R(\mathbf{0})\right)} \le C\|\nabla v\|_{\mathcal{L}^p\left(\omega,\Omega\right)}.$$

Using the results contained in [73] and [24] we have the following inequality,

$$\|v - v_\Omega\|_{\mathcal{L}^p\left(\omega,\Omega\right)} \le \left\|\widetilde{v} - v_\Omega\right\|_{\mathcal{L}^p\left(\omega,B_R(\mathbf{0})\right)} \le CR\,\rho\big(B_R(\mathbf{0})\big)^{\frac{1}{q}}\omega\big(B_R(\mathbf{0})\big)^{-\frac{1}{p}}\|\nabla v\|_{\mathcal{L}^p\left(\omega,\Omega\right)},$$

where v_Ω is the weighted mean of $\widetilde{v}$ over $B_R(\mathbf{0})$. Using the strong doubling property we have $\rho(B_R(\mathbf{0})) \le C\rho(\Omega)$, since $\rho(\Omega) \le \omega(B_R(\mathbf{0}))$ the above inequality becomes,

$$\|v - v_\Omega\|_{\mathcal{L}^p\left(\omega,\Omega\right)} \le CR\,\rho\big(\Omega\big)^{\frac{1}{q}}\omega\big(\Omega\big)^{-\frac{1}{p}}\|\nabla v\|_{\mathcal{L}^p\left(\omega,\Omega\right)}.$$

□

Corollary 2.16. *Let us consider a triangulation $\mathcal{T}$ that verifies the assumption* (2.9) *and Definition 2.1, $p \in (1, q]$, $\rho \in A_q(\mathbb{R}^2)$, $\omega \in A_p(\mathbb{R}^2)$ such that* (2.6) *holds. Then for every $T \in \mathcal{T}$ and $v \in \mathcal{W}^{2,2}(\omega, S_T)$ we have,*

$$\left\|\nabla(v - I^1_{\mathcal{N}})\right\|_{\mathcal{L}^q(\rho,T)} \le C\Big(\sigma_{\mathcal{T}}, \psi, C_{p,\omega}, C_{q,\omega}\Big)h_T\rho(S_T)^{\frac{1}{q}}\omega(S_T)^{-\frac{1}{p}}|v|_{\mathcal{W}^{2,2}(\omega,S_T)}.$$

Proof. Let consider again $T \in \mathcal{T}$ and denote $\mathbf{z}$ one of the vertices of T. We first notice that since ρ and ω verifies (2.6), we have thanks to Theorem 1.8 the following embedding,

$$\mathcal{W}^{2,p}_0(\omega, \Omega) \hookrightarrow \mathcal{W}^{1,q}(\rho, \Omega).$$

In view of the above embedding we have the following inequality,

$$\left\|\nabla(v - I^1_{\mathcal{N}})\right\|_{\mathcal{L}^q(\rho,T)} \le \left\|\nabla(v - Q^1_{\mathbf{z}}v)\right\|_{\mathcal{L}^q(\rho,T)} + \left\|\nabla(Q^1_{\mathbf{z}}v - I^1_{\mathcal{N}}v)\right\|_{\mathcal{L}^q(\rho,T)}.$$

Using (2.14) and Proposition 2.10 we know that,

$$\left\|\nabla(Q^1_{\mathbf{z}}v - I^1_{\mathcal{N}}v)\right\|_{\mathcal{L}^q(\rho,T)} \le C\left\|\nabla(v - Q^1_{\mathbf{z}}v)\right\|_{\mathcal{L}^q(\rho,T)}.$$

Combining the two last inequalities together with the fact that $\nabla Q^1_{\mathbf{z}}v = Q^0_{\mathbf{z}}\nabla v$ we get the following,

$$\left\|\nabla(v - I^1_{\mathcal{N}})\right\|_{\mathcal{L}^q(\rho,T)} \le \left\|\nabla v - Q^0_{\mathbf{z}}\nabla v\right\|_{\mathcal{L}^q(\rho,T)}.$$

Since $Q^0_{\mathbf{z}}\nabla v$ is the average value of ∇v on T we can conclude applying Proposition 2.15. $\square$

3

Finite Element Method

In this chapter I would like to study the application of the finite element method to domains presenting point singularities, in particular I'll focus my attention to the domain depicted in Figure 1.2; it is just a matter of calculation to generalize the idea here presented to domains with sharper corners. We have already discussed in the previous chapter that when the domain Ω presents a re-entrant corner we can not use the classical elliptic regularity which gives a nice shift triplet. It is well known that due to the absence of the shift triplet previously mentioned, the finite element method doesn't have the usual rate of convergence. This phenomena is not only observed in finite element methods, I redirect the reader interested in this phenomena for finite difference schemes to [89]. The above described phenomena it known in the engineering community as the **polluting effect** of the corner. The remainder of this chapter will be structured as follows: I'll discuss in more detail the consequences of the polluting effect of the corner in the context of the standard conforming finite element method and present the standard a priori error estimate analysis for the finite element method in domains presenting a re-entrant corner; next I'll introduce the penalty finite element method and perform an a priori error analysis for this method. Once the penalty finite element method is introduced, I'll show numerical evidence that the penalty finite element method, with a specific choice of the penalisation term, can retrieve optimal convergence with respect to the Sobolev space $\mathcal{W}^{\frac{5}{3},2}(\Omega)$ when the error is measured in the $\mathcal{L}^2(\Omega)$ norm. Next I'll show the impossibility of retrieving the optimal order of convergence using a weighted duality argument. It is important to mention that penalty finite element method is not the only or the most efficient method to deal with domain with re-entrant corners, but in my view it presents a very interesting case study for the difficulty of proving the result observed numerically and for its intimate connection with the Nitsche method that in recent years has become more and more popular. Let me redirect the reader interested in using finite element on domains with corners to more useful result. The hp-finite element method on domains with corners has been study by I. Babuska and B. Q. Guo, as far as I'm aware of they were also the first ones to realize that weighted Sobolev spaces are the correct setting where to study the error of finite element methods when dealing with domains presenting re-entrant corner, more information can be found in [57, 5, 58, 59]. I. Babuska and B. Q. Guo didn't limit themselves to the Poisson problem they also treated problems of elasticity and the Stokes problem, I redirect the reader interested in this kind of result to [56, 60, 61]. An other approach would be to use classical finite element methods together with mesh grading, [6, 8, 9, 10]. Furthermore it is also possible to construct finite element schemes solvable using multigrid methods, as shown in [19, 18, 21]. Last it is possible to consider the energy associated with the problem to obtain least squares finite element method that are optimal when dealing with singular solutions as shown in [33]. This last idea is different from what discussed in [44, 81, 80], involving an energy correcting term.

1 Conforming Finite Elements

In this section we will show an a priori error analysis for the conforming finite element method using standard techniques presented in [27]. In particular we will focus our attention to domains presenting a point singularity. To simplify the exposition of the idea here presented I'll focus my attention to the domain $\Omega \subset \mathbb{R}^2$ depicted in Figure 1.2. Let us consider once again the mesh $\mathcal{T}$ of the domain Ω and the discrete space V and $\overset{\circ}{V}$ introduced in (2.10). We consider our toy problem (1.16) with data $f \in \mathcal{L}^2(\Omega)$, whose solution from now on will be denoted as u_0. We now introduce the projection operator $\Pi_{\overset{\circ}{V}}$ that will give the finite element approximation of u_0 on the discrete space $\overset{\circ}{V}$, i.e.

$$\begin{gathered} \Pi_{\overset{\circ}{V}} : \mathcal{W}_0^{1,2}(\Omega) \to \overset{\circ}{V}, \\ \left(\nabla \Pi_{\overset{\circ}{V}} u, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} = \left(\nabla u, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} \qquad \forall v^h \in \overset{\circ}{V}. \end{gathered} \tag{3.1}$$

As usual we ask ourselves if this projection is well defined. In order to answer this question we can apply Hilbert projection theorem together with the following Lemma.

Lemma 1.1. *The discrete spaces V and $\overset{\circ}{V}$ are closed Hilbert linear subspaces of $\mathcal{W}^{1,2}(\Omega)$ and $\mathcal{W}_0^{1,2}(\Omega)$ respectively.*

Proof. I redirect the reader interested in this result to [27]. □

Proposition 1.2. *Let X be a Hilbert space and Y a closed linear subspace of X, then for every x in X there exists a unique y in Y such that,*

1. $\|x - y\|_X = \inf_{z \in Y} \|x - z\|_X$,
2. *the element y above introduced is uniquely characterized by, $(x, z)_X = (y, z)_X \; \forall z \in Y$.*

Proof. I redirect the reader interested in this result to [22], Chapter 5. □

Corollary 1.3. *The projection operator $\Pi_{\overset{\circ}{V}}$ is well defined.*

Proof. First we notice that the bilinear form $a(u, v) = (\nabla u, \nabla v)_{\mathcal{L}^2(\Omega)}$ is a scalar product on $\mathcal{W}_0^{1,2}(\Omega)$ because we have proven it is coercive in Corollary 2.11 and it is obviously continuous by Hölder inequality. Furthermore this scalar product is equivalent to $(\cdot, \cdot)_{\mathcal{W}^{1,2}(\Omega)}$. Since $\overset{\circ}{V}$ is a closed linear subspace of $\mathcal{W}_0^{1,2}(\Omega)$ we can apply the Hilbert projection and conclude. □

As the reader have probably already noticed such a projection is of limited use since we need to know a priori the solution u_0 to (1.16) in order to compute $\Pi_{\overset{\circ}{V}} u_0$. Using (1.16) we know that $(\nabla u_0, \nabla v^h)_{\mathcal{L}^2(\Omega)} = (f, v^h)_{\mathcal{L}^2(\Omega)}$, hence we can combine this together with (3.1) to characterize the projection $\Pi_{\overset{\circ}{V}} u_0$ with respect to the data f, i.e.

$$\left(\nabla \Pi_{\mathring{V}} u_0, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} = \left(f, v^h\right)_{\mathcal{L}^2(\Omega)} \qquad \forall v^h \in \mathring{V}. \tag{3.2}$$

This characterization of the projection (3.1) becomes very handy when we notice that since $\mathring{V}$ is a discrete space we can expand the function $v^h \in \mathring{V}$ in terms of the basis functions as done in (2.11). This combined with the linearity of the scalar product, yields the following set of equations:

$$\sum_{z \in \mathring{N}(\mathcal{T})} u_z^h \left(\nabla \phi_z, \nabla \phi_\eta\right)_{\mathcal{L}^2(\Omega)} = \left(f, \phi_\eta\right)_{\mathcal{L}^2(\Omega)} \qquad \forall \eta \in \mathring{N}(\mathcal{T}),$$

$$K\mathbf{U}^h = \mathbf{F}, \tag{3.3}$$

$$K_{z,\eta} = \left(\nabla \phi_z, \nabla \phi_\eta\right)_{\mathcal{L}^2(\Omega)} \qquad F_\eta = \left(f, \phi_\eta\right)_{\mathcal{L}^2(\Omega)} \qquad \forall z, \eta \in \mathring{N}(\mathcal{T}),$$

where K is usually called the stiffness matrix, $\mathbf{F}$ is usually called the load vector and $\mathbf{U}^h$ contains the value of $\Pi_{\mathring{V}} u_0$ at the nodes of the mesh. We will also call u_0^h the projection $\Pi_{\mathring{V}} u_0$ to stress the fact that it has been computed using (3.3). We now notice that (3.2) becomes,

$$a_0(u_0^h, v^h) = \left(\nabla u_0^h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} = \left(f, v^h\right)_{\mathcal{L}^2(\Omega)} \qquad \forall v^h \in \mathring{V}. \tag{3.4}$$

Proposition 1.4. *Let us be given a mesh $\mathcal{T}$ that verifies* (2.9) *and Definition 2.1. Let u_0 be the solution of* (1.16) *and u_0^h as above then the following energy error estimates holds,*

$$\left\|u_0 - u_0^h\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq C(\Omega) h^{\frac{2}{3}-\varepsilon} \|f\|_{\mathcal{L}^2(\Omega)}.$$

Proof. First of all I would like to warn the reader that the proof here included is slightly different from the standard one, because I wanted to make use of the interpolant constructed in the previous chapter. The classical proof will involve interpolation estimate in fractional Sobolev space, such as [38], and an argument similar to the one presented in [27]. We begin observing that,

$$\alpha \left\|u_0 - u_0^h\right\|^2_{\mathcal{W}^{1,2}(\Omega)} \overset{(1.17)}{\leq} a_0(u_0 - u_0^h, u_0 - u_0^h) = a_0(u_0 - u_0^h, u_0)$$

the last equality follows from **Galerkin orthogonality**, i.e. $a_0(u_0 - u_0^h, v^h) = (f, v^h)_{\mathcal{L}^2(\Omega)} - (f, v^h)_{\mathcal{L}^2(\Omega)} = 0$ for all $v \in \mathring{V}$.

Once again using Galerkin orthogonality we observe that $a_0(u_0 - u_0^h, I^1_{\mathcal{N}} u_0) = 0$ and therefore we have,

$$\alpha\left\|u_0-u_0^h\right\|^2_{\mathcal{W}^{1,2}(\Omega)} \overset{(1.17)}{\leq} a_0(u_0-u_0^h,u_0-I^1_{\mathcal{N}}u_0) \overset{H.}{\leq} \left\|u_0-u_0^h\right\|_{\mathcal{W}^{1,2}(\Omega)}\left\|u_0-I^1_{\mathcal{N}}u_0\right\|_{\mathcal{W}^{1,2}(\Omega)}$$
$$\alpha\left\|u_0-u_0^h\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq \left\|u_0-I^1_{\mathcal{N}}u_0\right\|_{\mathcal{W}^{1,2}(\Omega)}.$$

To conclude we observe that $\|u_0-I^1_{\mathcal{N}}u_0\|_{\mathcal{W}^{1,2}(\Omega)}\leq Ch^{\frac{2}{3}}\|f\|_{\mathcal{L}^2(\Omega)}$, this because of the fact that $\Big(H^1_0(\Omega),\mathcal{L}^2(|\mathbf{x}|^{\frac{2}{3}},\Omega),\mathcal{W}^{2,2}(|\mathbf{x}|^{\frac{2}{3}},\Omega)\Big)$ is a shift triplet for (1.16) together with Corollary 2.16 once we observe that,

$$\rho(S_T)=\int_{S_T}d\mathbf{x}\leq Ch_T^2\leq Ch^2,\qquad \omega(S_T)=\int_{S_T}|\mathbf{x}|^{\frac{2}{3}}d\mathbf{x}\leq Ch_T^{\frac{8}{3}}\leq Ch^{\frac{8}{3}}.$$

□

Now that we have an energy estimate for the error of the conforming finite element method I would like to proceed with the $\mathcal{L}^2(\Omega)$ a priori error analysis, but before I would like to make an observation:

$$\begin{aligned}\alpha\left\|u_0-u_0^h\right\|^2_{\mathcal{W}^{1,2}(\Omega)} &\overset{(1.17)}{\leq} \Big(\nabla(u_0-u_0^h),\nabla(u_0-u_0^h)\Big)_{\mathcal{L}^2(\Omega)}\\ &\overset{G.O.}{=} \Big(\nabla u_0,\nabla(u_0-u_0^h)\Big)=(f,u_0-u_0^h)_{\mathcal{L}^2(\Omega)},\\ \left\|u_0-u_0^h\right\|^2_{\mathcal{W}^{1,2}(\Omega)} &\overset{H.}{\leq} \frac{\|f\|_{\mathcal{L}^2(\Omega)}}{\alpha}\left\|u_0-u_0^h\right\|_{\mathcal{L}^2(\Omega)}.\end{aligned}\tag{3.5}$$

This last inequality combined with Proposition 1.4 tells us that the best possible error estimate I can obtain in the $\mathcal{L}^2(\Omega)$ norm is of the form,

$$\left\|u_0-u_0^h\right\|_{\mathcal{L}^2(\Omega)}\leq Ch^{\frac{4}{3}-\varepsilon}\|f\|_{\mathcal{L}^2(\Omega)}.$$

Indeed is possible to prove using the standard **Aubin-Nitsche** duality argument that the error in the $\mathcal{L}^2(\Omega)$ norm decays as described in the previous equation.

Theorem 1.5. *Given a mesh $\mathcal{T}$ that verifies* (2.9) *and Definition 2.1, let u_0 be the solution of* (1.16) *and u_0^h as in* (3.4). *Then the following energy error estimates holds,*

$$\left\|u_0-u_0^h\right\|_{\mathcal{L}^2(\Omega)}\leq C(\Omega)h^{\frac{4}{3}-\varepsilon}\|f\|_{\mathcal{L}^2(\Omega)}.$$

Proof. To begin we define the quantity $\varepsilon_h=u_0-u_0^h$ and consider the dual problem, find $w_0\in\mathcal{W}^{1,2}_0(\Omega)$ such that

$$a_0(w_0,v)=\Big(\nabla w_0,\nabla v\Big)_{\mathcal{L}^2(\Omega)}=\Big(\varepsilon_h,v\Big)_{\mathcal{L}^2(\Omega)}\qquad\forall v\in\mathcal{W}^{1,2}_0(\Omega).$$

Once again the existence and uniqueness of a solution for the dual problem follows from Lax-Milgram. Now we observe that if we take $v=\varepsilon_h$ then the above variational equation give us the following identity,

$$a_0(w_0, \varepsilon_h) = \Big(\nabla w_0, \nabla \varepsilon_h\Big)_{\mathcal{L}^2(\Omega)} = \Big(\varepsilon_h, \varepsilon_h\Big)_{\mathcal{L}^2(\Omega)} = \|\varepsilon_h\|^2_{\mathcal{L}^2(\Omega)}.$$

Once again we can observe that thanks to the Galerkin orthogonality we know that $a_0(I^1_{\mathcal{N}} w_0, \varepsilon_h) = a_0(I^1_{\mathcal{N}} w_0, u_0) - a_0(I^1_{\mathcal{N}} w_0, u^h_0) = (f, I^1_{\mathcal{N}} w_0)_{\mathcal{L}^2(\Omega)} - (f, I^1_{\mathcal{N}} w_0)_{\mathcal{L}^2(\Omega)} = 0$. We have:

$$a_0(w_0 - I^1_{\mathcal{N}} w_0, \varepsilon_h) = a_0(w_0, \varepsilon_h) = \Big(\nabla w_0, \nabla \varepsilon_h\Big)_{\mathcal{L}^2(\Omega)} = \Big(\varepsilon_h, \varepsilon_h\Big)_{\mathcal{L}^2(\Omega)} = \|\varepsilon_h\|^2_{\mathcal{L}^2(\Omega)}.$$

Using $\|w_0 - I^1_{\mathcal{N}} w_0\| \leq Ch^{\frac{2}{3}-\varepsilon}\|\varepsilon_h\|_{\mathcal{L}^2}(\Omega)$ we have the following inequality,

$$\begin{aligned}\|\varepsilon_h\|^2_{\mathcal{L}^2(\Omega)} &= a_0(w_0 - I^1_{\mathcal{N}} w_0, \varepsilon_h) \overset{H.}{\leq} \big|w_0 - I^1_{\mathcal{N}} w_0\big|_{\mathcal{W}^{1,2}(\Omega)} |\varepsilon_h|_{\mathcal{W}^{1,2}(\Omega)} \\ &\leq Ch^{\frac{2}{3}-\varepsilon}\|\varepsilon_h\|_{\mathcal{L}^2}(\Omega) h^{\frac{2}{3}-\varepsilon}\|f\|_{\mathcal{L}^2(\Omega)},\end{aligned}$$

dividing both sides by $\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}$ we yield,

$$\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)} \leq Ch^{\frac{4}{3}-2\varepsilon}\|f\|_{\mathcal{L}^2(\Omega)}.$$

□

We can observe that the numerical experiment presented in Figure 4.1 confirm the validity of the a priori error estimate proven in Proposition 1.4 and Theorem 1.5.

2 Penalty Finite Elements

In this section I would like to study a different flavour of conforming finite elements, known as penalty finite element method. The idea of penalty finite element methods is to solve (1.18) rather then (1.16), since as we have discussed in the first chapter the solution of (1.18) converges to (1.16) as $\varepsilon \to \infty$. Penalty finite element method enjoyed a brief moment of popularity in the early days of finite element methods: in particular estimate for the numerical error of penalty finite element method have been studied first by I. Babuska and J.P. Aubin in [7] and [3]. The estimate provided by Babuska and Aubin revealed sub-optimal in the cases studied by M. Utku and C. M. Carey [87]. A later explanation of the phenomena observed by M. Utku and C. M. Carey is presented by Zhong Sci in [85], under the assumption $u \in \mathcal{W}^{2,2}(\Omega)$. Furthermore error estimate assuming $u \in \mathcal{W}^{2,2}(\Omega)$ are presented by J.T. King and M. S. Serbin in [65, 66]. A similar problem has been studied also by J.W. Barrett and C. M. Elliot in [11] and by Z. Li. in [69]. Let me now introduce the discrete variational problem corresponding to (1.18),

$$\begin{gathered}\textit{Find } u^h_\varepsilon \in V \textit{ such that } \forall v \in V(\Omega)\ : \\ a_\varepsilon(u^h_\varepsilon, v) = \Big(\nabla u^h_\varepsilon, \nabla v\Big)_{\mathcal{L}^2(\Omega)} + \varepsilon^{-1}\Big(u^h_\varepsilon, v\Big)_{\mathcal{L}^2(\partial\Omega)} = \Big(f, v\Big)_{\mathcal{L}^2(\Omega)}\end{gathered} \tag{3.6}$$

We will begin presenting the argument introduced in [85] to obtain an error estimate in $\mathcal{W}^{1,2}(\Omega)$ for non smooth domain Ω. In particular for the rest of the chapter we wil focus our attention on the case where the bilinear form $a_\varepsilon(\cdot,\cdot)$ depends on the chosen mesh, i.e. $\varepsilon = h^{-\sigma}$. Furthermore for the remainder of this chapter we will assume the triangulation $\mathcal{T}$ associated with the discrete space V verifies (2.9) and Definition 2.1.

Remark 2.1. It is clear that the ε appearing in $\mathcal{W}^{s-\varepsilon,p}$ is different from the ε in (1.18). In order to avoid heavy notations from now on I will drop the dependence on ε when writing $\mathcal{W}^{s,p}$.

Lemma 2.2. *Let $u_0 \in \mathcal{W}^{\frac{5}{3},2}(\Omega)$, $u_\varepsilon^h \in V$, be respectively the solution of* (1.16) *and* (3.6), *with $f \in \mathcal{L}^2(\Omega)$ and $\varepsilon = h^{-\sigma}$. Then the following inequality holds:*

$$\left|u_0 - u_\varepsilon^h\right|^2_{\mathcal{W}^{1,2}(\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma + u_\varepsilon^h\right)^2 \le \left|u_0 - v^h\right|^2_{\mathcal{W}^{1,2}(\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma + v^h\right)^2, \tag{3.7}$$

for all $v^h \in V$.

Proof. Since u_ε^h minimizes the penalised energy (1.9) in V^h one has,

$$a_0(u_\varepsilon^h, u_\varepsilon^h) + h^{-\sigma}\int_{\partial\Omega}\left|u_\varepsilon^h\right|^2 ds - 2(f, u_\varepsilon^h)_{\mathcal{L}^2(\Omega)} \le a_0(v^h, v^h) + h^{-\sigma}\int_{\partial\Omega}\left|v^h\right|^2 ds - 2(f, v^h)_{\mathcal{L}^2(\Omega)}. \tag{3.8}$$

Moreover from the fact that u_0 is the solution of (1.16) in $\mathcal{W}_0^{1,2}$ we know that $a_0(u_0, v^h) - \int_{\partial\Omega}\frac{\partial u_0}{\partial n}v^h\, ds = (f, v^h)_{\mathcal{L}^2(\Omega)}$, and therefore the above inequality becomes,

$$\begin{aligned} &a_0(u_\varepsilon^h, u_\varepsilon^h) + h^{-\sigma}\int_{\partial\Omega}\left|u_\varepsilon^h\right|^2 ds - 2a_0(u_0, u_\varepsilon^h) - 2\int_{\partial\Omega}\frac{\partial u_0}{\partial n}u_\varepsilon^h\, ds \\ &\le a_0(v^h, v^h) + h^{-\sigma}\int_{\partial\Omega}\left|v^h\right|^2 ds - 2a_0(u_0, v^h) - 2\int_{\partial\Omega}\frac{\partial u_0}{\partial n}v^h\, ds. \end{aligned}$$

Rewriting the left hand side of (3.7) in terms of the bilinear form $a_0(\cdot,\cdot)$ we obtain,

$$\begin{aligned} &\left|u_0 - u_\varepsilon^h\right|^2_{\mathcal{W}^{1,2}(\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(h^\sigma\frac{\partial u_0}{\partial n} + u_\varepsilon^h\right)^2 ds \\ &= a_0(u_0, u_0) + h^\sigma\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}\right)^2 ds + a_0(u_\varepsilon^h, u_\varepsilon^h) + h^{-\sigma}\int_{\partial\Omega}\left(u_\varepsilon^h\right)^2 ds - 2a_0(u_\varepsilon^h, u_0) - 2\int_{\partial\Omega}\frac{\partial u_0}{\partial n}u_\varepsilon^h \\ &\le a_0(u_0, u_0) + h^\sigma\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}\right)^2 ds + a_0(v^h, v^h) + h^{-\sigma}\int_{\partial\Omega}\left(u_\varepsilon^h\right)^2 ds - 2a_0(v^h, u_0) - 2\int_{\partial\Omega}\frac{\partial u_0}{\partial n}v^h \\ &= \left|u_0 - v^h\right|^2_{\mathcal{W}^{1,2}(\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma + v^h\right)^2 ds \end{aligned} \tag{3.9}$$

□

The above lemma allows to prove an a priori error estimates for penalty finite element methods with respect to the $\mathcal{W}^{1,2}(\Omega)$ norm also in the context of non smooth domains.

Theorem 2.3. *Suppose we are in the hypothesis of the previous Lemma, then:*

$$\left\|u_0 - u_\varepsilon^h\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq C h^\mu \|f\|_{\mathcal{L}^2(\Omega)}$$

where $\mu = \min\left\{\frac{2}{3}, \frac{\sigma}{2}, \frac{7}{6} - \frac{\sigma}{2}\right\}$.

Proof. We take $v^h = I_{\mathcal{N}}^1$ and use Lemma 2.2 to obtain the following,

$$\begin{aligned} &\left|u_0 - u_\varepsilon^h\right|_{\mathcal{W}^{1,2}(\Omega)}^2 + h^{-\sigma} \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n} h^\sigma + u_\varepsilon^h\right)^2 ds \\ &\leq \left|u_0 - I_{\mathcal{N}}^1 u_0\right|_{\mathcal{W}^{1,2}(\Omega)}^2 + h^{-\sigma} \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n} h^\sigma + I_{\mathcal{N}}^1 u_0\right)^2 ds. \end{aligned} \tag{3.10}$$

Using the interpolation property $\|u_0 - I_{\mathcal{N}}^1 u_0\|_{\mathcal{W}^{1,2}(\Omega)} \leq C h^{\frac{2}{3}} \|f\|_{\mathcal{L}^2}(\Omega)$ presented in the previous section together with the trace inequality we obtain,

$$\begin{aligned} h^{-\sigma} \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n} h^\sigma + I_{\mathcal{N}}^1 u_0\right)^2 ds &\leq 2\left(h^\sigma \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n}\right)^2 ds + h^{-\sigma} \int_{\partial\Omega} \left(I_{\mathcal{N}}^1 u_0\right)^2 ds\right) \\ &\leq C(h^\sigma + h^{\frac{7}{3}-\sigma}) \|f\|_{\mathcal{L}^2(\Omega)}. \end{aligned}$$

Once again thanks to the interpolation property above we get,

$$\left|u_0 - u_\varepsilon^h\right|_{\mathcal{W}^{1,2}(\Omega)} \leq C h^{\frac{2}{3}} \|u_0\|_{\mathcal{W}^{\frac{5}{3},2}(\Omega)} \leq C h^{\frac{2}{3}} \|f\|_{\mathcal{L}^2(\Omega)}$$

and therefore,

$$h^{-\sigma} \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n} h^\sigma + u_\varepsilon^h\right)^2 ds + \left|u_0 - u_\varepsilon^h\right|_{\mathcal{W}^{1,2}(\Omega)} \leq C h^{2\mu} \|f\|_{\mathcal{L}^2(\Omega)}.$$

Last using the coercivity of the bilinear form $a_\varepsilon(\cdot,\cdot)$ we get,

$$\left\|u_0 - u_\varepsilon^h\right\|_{\mathcal{W}^{1,2}(\Omega)}^2 \leq C\left(h^{-\sigma} \int_{\partial\Omega} \left(u_\varepsilon^h\right)^2 ds + \left|u_0 - u_\varepsilon^h\right|_{\mathcal{W}^{1,2}(\Omega)}\right) \leq C h^{2\mu} \|f\|_{\mathcal{W}^{k,2}(\Omega)}.$$

□

Remark 2.4. For the previous theorem to work, we need to prove that we have a coercivity constant independent of ε,

$$a_\varepsilon(v,v) = \int_\Omega |\nabla v|^2 dx + \varepsilon^{-1} \int_{\partial\Omega} |v|^2 ds \geq \int_\Omega |\nabla v|^2 dx + \int_{\partial\Omega} |v|^2 ds \geq C(\Omega) \|v\|_{\mathcal{L}^2(\Omega)},$$

the last inequality was obtained using Poincaré-Friedrichs inequality, I redirect the reader interested in the proof of this result to [20], Chapter 10.

Remark 2.5. Starting with (3.8) we can get a different version of equation (3.9),

$$h^{-\sigma}\int_{\partial\Omega}\left(u_\varepsilon^h\right)^2\,ds \le \left|u_0 - v^h\right|^2_{\mathcal{W}^{1,2}(\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(v^h\right)^2\,ds + h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma\right)^2\,ds.$$

In fact taking the interpolant constructed in the previous chapter as v^h, we notice that $I^1_{\mathcal{N}}u_0$ will be null on $\partial\Omega$ as long as Ω is a polygon. Such phenomenon occurs because the triangulation exactly matches the boundary. Thanks to the above observation one gets on polygonal domains,

$$h^{-\sigma}\int_{\partial\Omega}\left(u_\varepsilon^h\right)^2\,ds \le h^\sigma\left\|\frac{\partial u_0}{\partial n}\right\|_{\mathcal{L}^2(\Omega)} \le Ch^\sigma\|f\|^2_{\mathcal{L}^2(\Omega)}.$$

In particular this modification to Lemma 2.2 improves the error estimate provided in Theorem 2.3.

$$\left\|u_0 - u_\varepsilon^h\right\|_{\mathcal{W}^{1,2}(\Omega)} \le Ch^\mu\|f\|_{\mathcal{L}^2(\Omega)}$$

where $\mu = \min\left\{\frac{2}{3},\frac{\sigma}{2}\right\}$. We notice that this error bound is also observed in the numerical experiments I carried out, as we can see from Figure 4.2.

We are now interested in the $\mathcal{L}^2(\Omega)$ error. To achieve this I'll present the duality trick by I. Babuska, first introduced in [7].

Theorem 2.6. *Assuming we are in the hypothesis of Lemma 2.2, we have:*

$$\left\|u_0 - u_\varepsilon^h\right\|_{\mathcal{L}^2(\Omega)} \le Ch^\alpha\|f\|_{\mathcal{L}^2(\Omega)},$$

where $\alpha = \min\left\{\mu+\frac{2}{3},\mu+\frac{7}{6}-\frac{\sigma}{2},\frac{7}{6},\sigma,\mu+\frac{\sigma}{2}\right\}$.

Proof. Let us introduce the quantity $\varepsilon_h := u_0 - u_\varepsilon^h$ and observe that by Lax-Milgram theorem there exists $w \in \mathcal{W}^{1,2}_0(\Omega)$ such that,

$$a_0(w,v) = (\varepsilon_h, v)_{\mathcal{L}^2(\Omega)} \qquad \forall v \in \mathcal{W}^{1,2}_0(\Omega),$$

in particular Corollary 3.7 tells us that $w \in \mathcal{W}^{\frac{5}{3}-\varepsilon,2}(\Omega)$ and

$$\|w\|_{\mathcal{W}^{\frac{5}{3}-\varepsilon,2}(\Omega)} \le \|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}.$$

Once again we use the fact that if u_0 solves the elliptic problem (1.16) and v belongs to $\mathcal{W}^{1,2}(\Omega)$ then,

$$a_0(u_0,v) = (f,v)_{\mathcal{L}^2(\Omega)} + \left(\frac{\partial u_0}{\partial n}, v\right)_{\mathcal{L}(\partial\Omega)}. \tag{3.11}$$

Since w is the solution of the dual problem and using (3.11) we know that,

$$a_0(\varepsilon_h, w) = (\varepsilon_h, \varepsilon_h)_{\mathcal{L}^2(\Omega)} + \left(\varepsilon_h, \frac{\partial w}{\partial n}\right)_{\mathcal{L}^2(\partial\Omega)},$$

$$a_0(\varepsilon_h, w) + a_0(\varepsilon_h, I^1_{\mathcal{N}}w - w) = h^{-\sigma}(\varepsilon_h, I^1_{\mathcal{N}}w)_{\mathcal{L}^2(\partial\Omega)} - \left(\frac{\partial u_0}{\partial n}, I^1_{\mathcal{N}}w\right)_{\mathcal{L}^2(\partial\Omega)},$$

where to obtain the last equality we have used the fact that $a_0(u^h_\varepsilon, v) = (f,v)_{\mathcal{L}^2(\Omega)} - h^{-\sigma}\int_{\partial\Omega} u^h_\varepsilon v\, ds$.

Now we can rewrite $(\varepsilon_h, \varepsilon_h)_{\mathcal{L}^2(\Omega)}$, as

$$\begin{aligned}\left|(\varepsilon_h, \varepsilon_h)_{\mathcal{L}^2(\Omega)}\right| \leq \left|a_0(\varepsilon_h, I^1_{\mathcal{N}}w - w)\right| + h^{-\sigma}\left|(\varepsilon_h, I^1_{\mathcal{N}}w)_{\mathcal{L}^2(\partial\Omega)}\right| + \left|\left(\frac{\partial u_0}{\partial n}, I^1_{\mathcal{N}}w\right)_{\mathcal{L}^2(\partial\Omega)}\right| \\ + \left|\left(\varepsilon_h, \frac{\partial w}{\partial n}\right)_{\mathcal{L}^2(\partial\Omega)}\right|.\end{aligned} \tag{3.12}$$

The first term in (3.12) can be controlled using the usual interpolation property and Theorem 2.3. We now consider the functional,

$$R(v) = a_0(u_0 - v, u_0 - v) + h^{-\sigma}\left(\frac{\partial u_0}{\partial n}h^\sigma + v, \frac{\partial u_0}{\partial n}h^\sigma + v\right)_{\mathcal{L}^2(\partial\Omega)}.$$

Such functional is minimized by u_0 and following the same reasoning presented in the proof of Theorem 2.3 one gets, $R(u^h_\varepsilon) \leq Ch^{2\mu}\|f\|_{\mathcal{L}^2(\Omega)}$. Therefore

$$h^{-\sigma}\left(\frac{\partial u_0}{\partial n}h^\sigma + u^h_\varepsilon, \frac{\partial u_0}{\partial n}h^\sigma + u^h_\varepsilon\right)_{\mathcal{L}^2(\partial\Omega)} \leq Ch^{2\mu}\|f\|^2_{\mathcal{L}^2(\Omega)}$$

$$\left(\frac{\partial u_0}{\partial n}h^\sigma + u^h_\varepsilon, \frac{\partial u_0}{\partial n}h^\sigma + u^h_\varepsilon\right)_{\mathcal{L}^2(\partial\Omega)} \leq Ch^{2\mu+\sigma}\|f\|^2_{\mathcal{L}^2(\Omega)},$$

$$\left(\frac{\partial u_0}{\partial n}h^\sigma, \frac{\partial u_0}{\partial n}h^\sigma\right)_{\mathcal{L}^2(\partial\Omega)} + 2\left(\frac{\partial u_0}{\partial n}h^\sigma, u^h_\varepsilon\right)_{\mathcal{L}^2(\partial\Omega)} + \left(u^h_\varepsilon, u^h_\varepsilon\right)_{\mathcal{L}^2(\partial\Omega)} \leq Ch^{2\mu+\sigma}\|f\|^2_{\mathcal{L}^2(\Omega)} \tag{3.13}$$

We now use Cauchy's inequality with a parameter β to observe that,

$$\left|\left(h^\sigma\frac{\partial u_0}{\partial n}, u^h_\varepsilon\right)_{\mathcal{L}^2(\partial\Omega)}\right| \leq \frac{h^{2\sigma}}{\beta}\left\|\frac{\partial u_0}{\partial n}\right\|^2 + \beta\left\|u^h_\varepsilon\right\|^2$$

Choosing $\beta = \frac{1}{2}$ then (3.13) becomes,

$$(u^h_\varepsilon, u^h_\varepsilon)_{\mathcal{L}^2(\partial\Omega)} \leq C\left(h^{2\mu+\sigma}\|f\|^2_{\mathcal{L}^2(\Omega)} + h^{2\sigma}\left(\frac{\partial u_0}{\partial n}, \frac{\partial u_0}{\partial n}\right)_{\mathcal{L}^2(\partial\Omega)}\right). \tag{3.14}$$

Combining (3.14), (3.12), Theorem 2.3 and the usual interpolation property we can conclude,

$$\left|(\varepsilon_h,\varepsilon_h)_{\mathcal{L}^2(\Omega)}\right| \le MCh^{\mu}\|f\|_{\mathcal{L}^2(\Omega)}h^{\frac{2}{3}}\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)} + Ch^{-\sigma}\left(h^{\mu+\frac{\sigma}{2}}\|f\|_{\mathcal{L}^2(\Omega)} + h^{\sigma}\left\|\frac{\partial u_0}{\partial n}\right\|_{\mathcal{L}^2(\partial\Omega)}\right)h^{\frac{7}{6}}\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}$$

$$+C\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}h^{\frac{7}{6}}\left\|\frac{\partial u_0}{\partial n}\right\|_{\mathcal{L}^2(\partial\Omega)} + C\left(h^{\mu+\frac{\sigma}{2}}\|f\|_{\mathcal{L}^2(\Omega)} + h^{\sigma}\left\|\frac{\partial u_0}{\partial n}\right\|_{\mathcal{L}^2(\partial\Omega)}\right)\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}$$
$$\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)} \le MCh^{\alpha}\|f\|_{\mathcal{L}^2(\Omega)}.$$

Since we are using the continuity of $a_0(\cdot,\cdot)$, M doesn't explode as $\varepsilon \to 0$. □

Remark 2.7. Since $\gamma_0(I^1_N w) = 0$ then (3.12) becomes,

$$\left|(\varepsilon_h,\varepsilon_h)_{\mathcal{L}^2(\Omega)}\right| \le \left|a_0(\varepsilon_h, I^1_N w - w)\right| + \left|\left(\varepsilon_h, \frac{\partial w}{\partial n}\right)_{\mathcal{L}^2(\partial\Omega)}\right|,$$

following the same reasoning presented in the proof of Theorem 2.6 this yields,

$$\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)} \le MCh^{\alpha}\|f\|_{\mathcal{L}^2(\Omega)},$$

where $\alpha = \left\{\mu + \frac{2}{3}, \sigma, \mu + \frac{\sigma}{2}\right\}$.
If we combine this comment with Remark 2.5 we get, $\alpha = \left\{\frac{4}{3}, \frac{2}{3} + \frac{\sigma}{2}, \sigma\right\}$.

A careful reader might notice that for both the error estimates in $\mathcal{W}^{1,2}(\Omega)$ and in $\mathcal{L}^2(\Omega)$ we first presented a general error estimates and then in a sequent remark noticed how the fact that $\gamma_0(I^1_N w) = 0$ affects the error estimates just proven. This because the interpolant presented in the previous chapter is a particular case of the construction made in [79], and we want to leave the result in a usable form also for future generalisation with other form of the interpolant presented in [79].
We notice that the numerical experiments presented in Figure 4.3 confirm the a priori estimates presented in Remark 2.7, but also show an interesting phenomena i.e. the estimate presented in the above Remark are **suboptimal** for $\sigma = \frac{5}{3}$. The next sections will be devoted to observation regarding this behaviour.

3 Weighted Duality Argument

The originally intended name for this chapter was "68 ways not to prove the numerical phenomena observed at the end of the previous section", but instead of showing to the reader many unsuccessful proofs I opted to argue why it is not possible using a duality argument to retrieve optimal rate of convergence in $\mathcal{L}^2(\Omega)$ when $\sigma = \frac{5}{3}$. First I would like to show the reader why an argument like the one in (3.5) does not apply in the context of penalty finite element method and therefore there might still be hope of retrieving a $\frac{5}{3}$ rate of convergence in $\mathcal{L}^2(\Omega)$ even if in $\mathcal{W}^{1,2}(\Omega)$ we only have a $\frac{2}{3}$ rate of convergence. We know from Remark 2.4 that we have for penalty finite element methods a coercivity constant that is independent from h and therefore we can write,

$$\begin{aligned}
\frac{\left\|u_0-u_\varepsilon^h\right\|^2_{\mathcal{W}^{1,2}(\Omega)}}{C(\Omega)} &\leq a_\varepsilon(u_0-u_\varepsilon^h,u_0-u_\varepsilon^h) \leq a_\varepsilon(u_0,u_0-u_\varepsilon^h)-a_\varepsilon(u_\varepsilon^h,u_0-u_\varepsilon^h)\\
&\leq a_\varepsilon(u_0,u_0-u_\varepsilon^h)-a_\varepsilon(u_\varepsilon^h,u_0)+a_\varepsilon(u_\varepsilon^h,u_\varepsilon^h)\\
&\leq a_\varepsilon(u_0,u_0-u_\varepsilon^h)-(f,u_\varepsilon^h)_{\mathcal{L}^2(\Omega)}-(\partial_n u_0,u_\varepsilon^h)_{\mathcal{L}^2(\partial\Omega)}+(f,u_\varepsilon^h)_{\mathcal{L}^2(\Omega)}\\
&\leq a_\varepsilon(u_0,u_0-u_\varepsilon^h)-(\partial_n u_0,u_\varepsilon^h)_{\mathcal{L}^2(\partial\Omega)}\\
&\leq (f,u_0-u_\varepsilon^h)_{\mathcal{L}^2(\Omega)}-2(\partial_n u_0,u_\varepsilon^h)_{\mathcal{L}^2(\partial\Omega)}\\
&\leq \|f\|_{\mathcal{L}^2(\Omega)}\left\|u_0-u_\varepsilon^h\right\|_{\mathcal{L}^2(\Omega)}-2(\partial_n u_0,u_\varepsilon^h)_{\mathcal{L}^2(\partial\Omega)},
\end{aligned}$$

therefore we have the following inequality,

$$\left\|u_0-u_\varepsilon^h\right\|^2_{\mathcal{W}^{1,2}(\Omega)} \leq C(\Omega)\|f\|_{\mathcal{L}^2(\Omega)}\left\|u_0-u_\varepsilon^h\right\|_{\mathcal{L}^2(\Omega)}-C(\Omega)2(\partial_n u_0,u_\varepsilon^h)_{\mathcal{L}^2(\partial\Omega)}.$$

Since there is no guarantee that $C(\Omega)2(\partial_n u_0,u_\varepsilon^h)_{\mathcal{L}^2(\partial\Omega)}$ is positive, we still can retrieve a $\frac{5}{3}$ rate of convergence in $\mathcal{L}^2(\Omega)$ even if in $\mathcal{W}^{1,2}(\Omega)$ we only have a $\frac{2}{3}$ rate of convergence. Going back to the proof of the $\mathcal{L}^2(\Omega)$ error estimate for penalty finite elements presented in the previous section we notice that a key step of both Babuska and Aubin-Nitsche duality arguments is to bound from above the quantity, $|a_0(\varepsilon_h,I^1_\mathcal{N}w-w)|$. Furthermore in both proof of Theorem 1.5 and Theorem 2.6 we showed that the best possible estimate we can obtain is of the form,

$$\left|a_0(\varepsilon_h,I^1_\mathcal{N}w-w)\right| \leq Ch^{\frac{4}{3}}\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)},$$

while ideally we are searching for an error estimate of the following form when $\sigma=\frac{5}{3}$,

$$\left|a_0(\varepsilon_h,I^1_\mathcal{N}w-w)\right| \leq Ch^{\frac{5}{3}}\|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}.$$

Once again this mismatch between the error estimate that we have and the one that we want is due to the **polluting effect** of the corner. In fact since u_0,w_0 only lives in $\mathcal{W}^{\frac{5}{3},2}(\Omega)$ we can only have error estimates of the form,

$$\begin{aligned}
&\left\|u_0-u_0^h\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq Ch^{\frac{2}{3}}\|f\|_{\mathcal{L}^2(\Omega)} \qquad \left\|w_0-I^1_\mathcal{N}w_0\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq Ch^{\frac{2}{3}}\|\varepsilon\|_{\mathcal{L}^2(\Omega)},\\
&\left\|u_0-u_\varepsilon^h\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq Ch^{\frac{2}{3}}\|f\|_{\mathcal{L}^2(\Omega)} \qquad \left\|w-I^1_\mathcal{N}w\right\|_{\mathcal{W}^{1,2}(\Omega)} \leq Ch^{\frac{2}{3}}\|\varepsilon\|_{\mathcal{L}^2(\Omega)}.
\end{aligned}$$

respectively for the Aubin-Nitsche duality trick and the Babuska duality trick. Now the reader might think, as the writer did when first dealing with this problem, that since the $\frac{4}{3}$ rate of convergence is caused by the fact that $u_0,w_0\in\mathcal{W}^{\frac{5}{3},2}(\Omega)$ a good way of solving this problem is to take full advantage of the regularity results we have proven in the previous section for Muckenhoupt weighted Sobolev spaces and domain with point singularities. In particular the key idea here is to evaluate the error norm of ε_h in a $\mathcal{W}^{2,2}(|\mathbf{x}|^{-\gamma},\Omega)$ while we evaluate the error norm of $w_0 - I^1_{\mathcal{N}} w_0$ in a $\mathcal{W}^{2,2}(|\mathbf{x}|^{\gamma},\Omega)$, i.e.

$$\begin{aligned} a_0(\varepsilon_h, I^1_{\mathcal{N}} w - w) &= \int_\Omega \nabla\varepsilon_h \nabla\big(w_0 - I^1_{\mathcal{N}} w_0\big)\, d\mathbf{x} = \int_\Omega \nabla\varepsilon_h |\mathbf{x}|^{\gamma} \nabla\big(w_0 - I^1_{\mathcal{N}} w_0\big)|\mathbf{x}|^{-\gamma}\, d\mathbf{x} \\ &\overset{H.}{\le} |\varepsilon_h|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)} \big|w - I^1_{\mathcal{N}} w\big|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma},\Omega)}. \end{aligned} \tag{3.15}$$

At this point we can use Corollary 2.16 to provide estimates for $|w - I^1_{\mathcal{N}} w|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)}$. Furthermore we will assume that u^h_ε behaves as the interpolant $I^1_{\mathcal{N}} u_0$ in $\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma},\Omega)$. In particular once we observe that for weights of the form $\omega(\mathbf{x}) = |\mathbf{x}|^\gamma$ the following chain of inequality holds,

$$\omega(S_T) = \int_{S_T} |\mathbf{x}|^\gamma\, d\mathbf{x} \le \int_{B_{r(\sigma_{\mathcal{T}},h_T)}(\mathbf{0})} |\mathbf{x}|^\gamma\, d\mathbf{x} \le C(\sigma_{\mathcal{T}}) h^{\gamma+2}$$

then Corollary 2.16 yields the following,

$$\begin{aligned} |\varepsilon_h|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)} &\approx \big|u_0 - I^1_{\mathcal{N}} u_0\big|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)} \le C(\Omega) h h^{\frac{\gamma}{2}+1} h^{-\frac{4}{3}} \|f\|_{\mathcal{L}^2(|\mathbf{x}|^{-\gamma},\Omega)} \\ &\le C(\Omega) h^{\frac{\gamma}{2}+\frac{2}{3}} \|f\|_{\mathcal{L}^2(|\mathbf{x}|^{-\gamma},\Omega)}, \\ \big|w_0 - I^1_{\mathcal{N}} w_0\big|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma},\Omega)} &\le C(\Omega) h h^{-\frac{\gamma}{2}+1} h^{-\frac{4}{3}} \|\varepsilon_h\|_{\mathcal{L}^2(|\mathbf{x}|^{\gamma},\Omega)} \le C(\Omega) h^{\frac{2}{3}-\frac{\gamma}{2}} \|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}, \end{aligned} \tag{3.16}$$

and therefore (3.15) gives us for all positive $\gamma \in A_2(\mathbb{R}^2)$ the estimate that comes next, i.e.

$$a_0(\varepsilon_h, I^1_{\mathcal{N}} w - w) \overset{H.}{\le} |\varepsilon_h|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma},\Omega)} \big|w - I^1_{\mathcal{N}} w\big|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)} \le C(\Omega) h^{\frac{4}{3}} \|\varepsilon_h\|_{\mathcal{L}^2(\Omega)} \|f\|_{\mathcal{L}^2(|\mathbf{x}|^{-\gamma},\Omega)}.$$

In conclusion the reader can notice that even if we exploit the full regularity of the solution u_0, in Muckenhoupt weighted Sobolev spaces, it is not possible to retrieve by a duality trick the desired estimate, i.e. $|a_0(\varepsilon_h, I^1_{\mathcal{N}} w - w)| \le C h^{\frac{5}{3}} \|\varepsilon_h\|_{\mathcal{L}^2(\Omega)}$. We make the reader aware of the fact that in order to prove estimate (3.16), we used the natural embedding,

$$\begin{aligned} &\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega) \hookrightarrow \mathcal{W}^{1,2}(\Omega), \\ &\|\cdot\|_{\mathcal{W}^{1,2}(\Omega)} \le \|\cdot\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)}. \end{aligned}$$

A careful reader might notice that if in (3.15) we can not use the Holder inequality with Holder exponents p, q different from 2, because to verify the hypothesis of Corollary we need $p \in (1, q]$ and $q \in (1, p]$.

4 Petrov-Galerkin method

It turns out that the idea of studying the bilinear form $a_0(\cdot,\cdot)$ as mapping from two different Muckenhoupt weighted Sobolev spaces to $\mathbb{R}$, i.e.

$$a_0 : \mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma}, \Omega) \times \mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma}, \Omega) \to \mathbb{R}$$

is the correct way to obtain error estimates with respect to the $\mathcal{L}^2(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ norm and $\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ norm. In fact the objective of this section is to present an a priori analysis with respect to the $\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ norm, following similar ideas to the one presented in [34] and [8]. The key tool of this section will be the Brezzi-Necas-Babuska theorem and a decomposition lemma for $\mathcal{L}^2(|x|^{\gamma}, \Omega)$,

Theorem 4.1 (Brezzi-Necas-Babuska). *Let X and Y be two Hilbert spaces, $a : X \times Y \to \mathbb{R}$ a bilinear form and consider the variational problem, find $u \in X$ such that*

$$a(u, v) = \big(f, v\big)_Y \qquad \forall v \in Y,$$

where $f \in Y^*$. The above variational problem is well-posed if and only if,

1. *$\forall v \in Y$ if $a(u, v) = 0\ \forall u \in X$ then $v \equiv 0$,*
2. *there exists $\alpha > 0$ such that*

$$\inf_{u \in X} \sup_{v \in Y} \frac{a(u, v)}{\|u\|_X \|v\|_Y} \geq \alpha. \tag{3.17}$$

Furthermore the following stability estimate $|u|_X \leq \frac{1}{\alpha}\|f\|_{Y^*}$ holds.

Proof. I redirect the reader interested in the proof of this result to [46], Chapter 2. □

Remark 4.2. Condition 1. in the previous theorem can be swapped with,

$$\inf_{v \in Y} \sup_{u \in X} \frac{a(u, v)}{\|u\|_X \|v\|_Y} \geq \overline{\alpha}. \tag{3.18}$$

In fact, we fix $v \in Y$ and observe that above inf-sup condition tells us that there exits one $u^* \in X$ such that $a(u, v) \geq \alpha \|u\|_X \|v\|_Y$ and therefore $a(u, v) = 0\ \forall u \in X$ implies also $a(u^*, v) = 0$, i.e.

$$0 \geq a(u^*, v) \geq \alpha \|u^*\|_X \|v\|_Y \Rightarrow v \equiv 0.$$

Lemma 4.3. *If $\gamma \in (-2, 2)$ then the following decomposition for the space $\mathcal{L}^2(|x|^{\gamma}, \Omega)$ holds,*

$$\Big[\mathcal{L}^2(|x|^{\gamma}, \Omega)\Big]^d = \Big(\nabla \mathcal{W}_0^{1,2}(|x|^{\gamma}, \Omega)\Big) \bigoplus \Big(\nabla \mathcal{W}_0^{1,2}(|\mathbf{x}|^{-\gamma}, \Omega)\Big)^{\perp}.$$

Proof. This result is a consequence of the fact that for $\gamma \in (-2, 2)$ both $|\mathbf{x}|^{-\gamma}$ and $|\mathbf{x}|^{\gamma}$ are Muckenhoupt weights and therefore all the Muckenhoupt weighted spaces mentioned in this Lemma are Hilbert. In fact the key argument to prove this result is the well posedness of the mixed problem,

$$\textit{Given } \mathbf{q} \in \big[\mathcal{L}^2(|\mathbf{x}|^{\gamma}, \Omega)\big]^n, \textit{ find } (\sigma, z) \in \mathcal{L}^2(|x|^{\gamma}, \Omega) \times \mathcal{W}_0^{1,2}(|\mathbf{x}|^{\gamma}, \Omega) \textit{ s.t.}$$

$$\begin{cases} (\sigma, \tau)_{\mathcal{L}^2(\Omega)} + (\nabla z, \tau)_{\mathcal{L}^2(\Omega)} = (\mathbf{q}, \tau)_{\mathcal{L}^2(\Omega)} \\ (\nabla w, \sigma)_{\mathcal{L}^2(\Omega)} = 0 \end{cases}, \tag{3.19}$$

which can be proven using the Banach-Brezzi-Babuska Theorem. I redirect the reader interested in a detailed proof of this result to [34], while more information on the BBB Theorem can be found in [14]. ☐

Corollary 4.4. *Given a data $f \in \mathcal{L}^2(|\mathbf{x}|^{-\gamma}, \Omega)$ the following variational problem,*

$$find\ u_0 \in \mathcal{W}_0^{1,2}(|x|^{\gamma}, \Omega)\ such\ that\ a_0(u_0, v) = (f, v)_{\mathcal{L}^2(\Omega)}\ for\ all\ v \in \mathcal{W}_0^{1,2}(|x|^{-\gamma}, \Omega),$$

is well-posed and we have the following stability estimate,

$$\|u\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma}, \Omega)} \leq C\|f\|_{\mathcal{L}^2(|\mathbf{x}|^{-\gamma}, \Omega)}.$$

Proof. Let $v \in \mathcal{W}_0^{1,2}(|x|^{-\gamma}, \Omega)$ and consider $q = |\mathbf{x}|^{\gamma}\nabla v$ which by construction belongs to $\mathcal{L}^2(|\mathbf{x}|^{\gamma}, \Omega)$. Using Lemma 4.3 we can find $\sigma_v \in \mathcal{L}^2(|\mathbf{x}|^{\gamma}, \Omega)$ and $z_v \in \mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma}, \Omega)$ such that,

$$(\nabla z_v, \nabla v)_{\mathcal{L}^2(\Omega)} = (\mathbf{q}, \nabla v)_{\mathcal{L}^2(\Omega)} - (\sigma_v, \nabla v)_{\mathcal{L}^2(\Omega)} = (\mathbf{q}, \nabla v)_{\mathcal{L}^2(\Omega)} = \|v\|^2_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma}, \Omega)}.$$

The above equality together with the fact that $\|z_v\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma}, \Omega)} \leq C\|q\|_{\mathcal{L}^2(|\mathbf{x}|^{\gamma}, \Omega)} = \|v\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma}, \Omega)}$, yields:

$$\frac{(\nabla z_v, \nabla v)_{\Omega}}{\|z_v\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\gamma}, \Omega)}} \geq \frac{(\nabla z_v, \nabla v)_{\Omega}}{C\|v\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma}, \Omega)}} = \frac{\|v\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma}, \Omega)}}{C}.$$

Therefore (3.17) is verified and by a similar argument also (3.18) is verified. We conclude applying Theorem 4.1. ☐

Remark 4.5. Now given $T \in \mathcal{T}$ we define the quantity, $\overline{r}_T = \max_{\mathbf{x} \in T} d(\mathbf{x}, \mathbf{0})$, we will need the fact that the discrete norm $\|v^h\|_{h,\gamma}$ defined as, $\|v^h\|^2_{h,\gamma} = \sum_{T \in \mathcal{T}} \overline{r}_T^{\gamma} \|v^h\|^2_{\mathcal{L}^2(T)}$ is equivalent to the continuous one, i.e.

$$c\|v^h\|^2_{h,\gamma} \leq \|v^h\|^2_{\mathcal{L}^2(|\mathbf{x}|^{\gamma}, \Omega)} \leq C\|v^h\|^2_{h,\gamma}. \tag{3.20}$$

I will only consider the case $\gamma \geq 0$ because the case $\gamma < 0$ can be proved analogously. I fix the notation S_{T_0} to denote the diamond of simplexes that have $\mathbf{0}$ as vertex. First I observe that one of the inequalities in (3.20) comes for free while the other one has a more involved proof. In particular we observe that there exist $C_1(\sigma), C_2(\sigma) > 0$ such that for all $T \in \mathcal{T} \backslash S_{T_0}$,

$$C_1(\sigma_{\mathcal{T}})|\mathbf{x}|^{\gamma} \leq \overline{r}_T \leq C_2(\sigma_{\mathcal{T}})|\mathbf{x}|^{\gamma} \qquad \forall \mathbf{x} \in T.$$

Therefore for the simplexes T in $\mathcal{T} \backslash S_{T_0}$ we have have a bound of the form,

$$C(\sigma_{\mathcal{T}})\overline{r}_T\|u_0^h\|^2_{\mathcal{L}^2(T)} \leq \left\||\mathbf{x}|^{\frac{\gamma}{2}} u_0^h\right\|^2_{\mathcal{L}^2(T)}.$$

We aim at proving a similar bound also for $T \subset S_{T_0}$ and we will do so by using a scaling argument. Let $\hat{T}$ be the usual reference element, $F_T : \hat{T} \to T$ the usual affine mapping from $\hat{T}$

to T, $\hat{u}_h = u_0^h \circ F_T$ and $\hat{\mathbf{x}}_0 = F_T^{-1}(\mathbf{0})$, then by the shape regularity assumption we know there exists a $C_3 > 0$ such that $|F_T \hat{\mathbf{x}}|^\gamma \geq C_3(\sigma_{\mathcal{T}}) h_T |\hat{\mathbf{x}}|$ and therefore,

$$\left\| |\mathbf{x}|^{\frac{\gamma}{2}} u_0^h \right\|^2_{\mathcal{L}^2(T)} = \int |\mathbf{x}|^\gamma \left|u_0^h\right|^2 d\mathbf{x} = \frac{|T|}{\left|\hat{T}\right|} \int_{\hat{T}} |F_T \hat{\mathbf{x}}| |\hat{u}_h|^2 \, d\hat{\mathbf{x}} \geq C_3(\sigma_{\mathcal{T}}) h_T^\gamma \frac{|T|}{\left|\hat{T}\right|} \int_{\hat{T}} |\hat{\mathbf{x}}| |\hat{u}_h|^2 \, \hat{\mathbf{x}}. \quad (3.21)$$

We introduce $\Delta > 0$ and introduce the sub reference element,

$$\hat{T}_\Delta = \left\{ \mathbf{x} \in \hat{T} \, : \, d(\mathbf{x}, \mathbf{0}) > \Delta \right\},$$

and we notice that for Δ sufficiently small we have,

$$\int_{\hat{T}} |\hat{\mathbf{x}}| \hat{u}_h^2 \, \hat{\mathbf{x}} \geq \Delta^\gamma \|\hat{u}_h\|^2_{\mathcal{L}^2(\hat{T}_\Delta)}, \qquad \left|\hat{T}_\Delta\right| \geq \left(1 - \mathcal{O}(\Delta^2)\right) \left|\hat{T}\right|, \qquad \|\hat{u}_h\|^2_{\mathcal{L}^2(\hat{T}_\Delta)} \leq C(\Delta) \|\hat{u}_h\|^2_{\mathcal{L}^2(\hat{T})}.$$

Combining these together with (3.21) we get, the following,

$$\left\| |\mathbf{x}|^{\frac{\gamma}{2}} u_0^h \right\|^2_{\mathcal{L}^2(T)} \geq C_3(\sigma_{\mathcal{T}}) h_T^\gamma \frac{|T|}{\left|\hat{T}\right|} \int_{\hat{T}} |\hat{\mathbf{x}}| \hat{u}_h^2 \, \hat{\mathbf{x}} \geq C_3(\sigma_{\mathcal{T}}) \Delta^\gamma h_T^\gamma \frac{|T|}{\left|\hat{T}\right|} \|\hat{u}_h\|^2_{\mathcal{L}^2(\hat{T}_\Delta)} \geq C(\sigma_{\mathcal{T}}, \Delta) \overline{r}_T^\gamma \|\hat{u}_h\|^2_{\mathcal{L}^2(T)}.$$

The key idea here is that since Δ was chosen in the reference domain it is independent from h_T and it doesn't need to go to zero as h_T goes to zero.

Since we have the norm equivalence (3.20) we will resort to a discrete version of Lemma 4.3, in order to prove the well posedness of the discrete variational problem (3.4).

Lemma 4.6. *Consider $\gamma \in (-2, 2)$ and the discrete functional space,*

$$\mathcal{M}^0 = \left\{ \mathbf{q}_h \in \left[\mathcal{L}^2(\Omega)\right]^d \, : \, \mathbf{q}_h\big|_T \in \mathbb{P}_0(T) \; \forall T \in \mathcal{T} \right\}.$$

Then the following decomposition holds,

$$\mathcal{M}^0 = \left(\nabla \mathring{V}\right) \bigoplus \left(\nabla \mathring{V}\right)^\perp.$$

Proof. Once again I redirect the reader interested in the proof of this result to [34], but essentially the argument there presented is the well posedness of the discrete version of (3.19). □

Theorem 4.7. *The discrete variational problem* (3.4) *is well posed and we have the following error estimate,*

$$\left\|u_0 - u_0^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}^\gamma|, \Omega)} \leq C \inf_{v^h \in \mathring{V}} \left\|u_0 - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma, \Omega)} \quad (3.22)$$

Proof. We consider $v^h \in \mathring{V}$ and define the function $q_h\big|_T = (\overline{r}_T)^{-\gamma} \nabla v_h\big|_T$ for all $T \in \mathcal{T}$. Using the previous lemma we can find $(\sigma_h, z_h) \in \mathcal{M}^0 \times \mathring{V}$ such that,

$$\begin{aligned}\|\nabla z_h\|_{h,\gamma} &\le 2\|\mathbf{q}_h\|_{h,\gamma} = 2\|\nabla v\|_{h,\gamma},\\ \left(\nabla z_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} &= \left(\mathbf{q}_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} - \left(\sigma_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)}\\ &= \left(\nabla z_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} = \left(\mathbf{q}_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)} = \left\|\nabla v^h\right\|^2_{h,-\gamma}.\end{aligned}$$

Therefore for all $v^h \in \overset{\circ}{V}$ we have that, $\sup_{w_h\in\overset{\circ}{V}} \frac{\left(\nabla w_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)}}{\|w_h\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)}} \ge \frac{\left(\nabla z_h, \nabla v^h\right)_{\mathcal{L}^2(\Omega)}}{\|z_h\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)}} \overset{(3.20)}{\ge} C\left\|\nabla v^h\right\|_{h,-\gamma}$. The other inf-sup condition can be proven in an analogous manner and therefore the problem is well posed. We prove (3.22), observing that for all $(u_0^h - v^h) \in \mathcal{W}^{1,2}(|\mathbf{x}|^\gamma, \Omega)$

$$C\left\|u_0^h - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} \le \sup_{w_h\in\overset{\circ}{V}} \frac{a_0(u_0^h - v^h, w_h)}{\|w_h\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)}} = \sup_{w_h\in\overset{\circ}{V}} \frac{a_0(u - v^h, w_h)}{\|w_h\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)}},$$

where to obtain the last equality we used the Galerkin orthogonality, i.e. $a(u-u_0^h, w_h) = 0$. We notice that the bilinear form $a_0(\cdot,\cdot)$ is continuous by Hölder inequality and therefore we have,

$$C\left\|u_0^h - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} \le \sup_{w_h\in\overset{\circ}{V}} \frac{a_0(u - v^h, w_h)}{\|w_h\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)}} \le \left\|u - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)}.$$

Now using the triangular inequality we have,

$$\left\|u - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} \le \left\|u_0 - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} + \left\|v^h - u_0^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} \le \left(1 + \frac{1}{C}\right)\left\|u - v^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)},$$

since the above inequality holds for all $v \in \overset{\circ}{V}$ we can take the inf and conclude. □

Corollary 4.8. *The following error bound holds for the solution of the discrete variational problem* (3.4)*,*

$$\left\|u_0 - u_0^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}},\Omega)} \le Ch\|f\|_{\mathcal{L}^2(\Omega)}. \tag{3.23}$$

Proof. We simply apply (3.22) together with the fact that from the previous chapter we know that,

$$\left\|u_0 - I_{\mathcal{N}}^1 u_0\right\|_{\mathcal{W}^{1,2}(|x|^{\frac{2}{3}},\Omega)} \le Ch\|u_0\|_{\mathcal{W}^{2,2}(|x|^{\frac{2}{3}},\Omega)} \le Ch\|f\|_{\mathcal{L}^2(|x|^{\frac{2}{3}},\Omega)} \le Ch\|f\|_{\mathcal{L}^2(\Omega)}.$$

□

We can observe that the numerical experiment presented in Figure 4.4 confirms the numerical estimates presented above. The most natural question the reader might ask now is if we can prove an error estimate similar to (3.23) also for the penalty finite elements method. In order to achieve this result we need to prove the weighted counterpart of Lemma 2.2. To do this we notice that even if in a Petrov-Galerkin setting we don't have an energy minimization prospective we can retrieve this point of view in the discrete setting. In particular the key idea is that u_ε^h not only minimizes $J_\varepsilon(v)$ in V. In fact we observe that when we proved (3.20) we also showed that the scalar product $(\cdot,\cdot)_{h,\gamma}$ is equivalent to the scalar product $(\cdot,\cdot)_{\mathcal{W}^{1,p}(|\mathbf{x}|^\gamma,\Omega)}$ and therefore the solution of the variational problem,

$$\left(\nabla u_\varepsilon^h, \nabla v^h\right)_{h,\gamma} + h^{-\sigma}(u_\varepsilon^h, v^h)_{\partial\Omega,h,\gamma} = (f, v^h)_{h,\gamma} \,. \tag{3.24}$$

minimizes the following energy functional with in V,

$$J_{\varepsilon,\gamma}(v^h) = -\frac{1}{2}\left(\nabla v^h, \nabla v^h\right)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} + h^{-\sigma}(v^h, v^h)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\partial\Omega)} - (f, v^h)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} \,, \tag{3.25}$$

provided that we are working in the Hilbert spaces setting which is a consequence of asking $\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)$ to be Muckenhoupt weighted Sobolev space, i.e. $\gamma \in (-2,2)$. If we consider (3.6) and instead of v we consider a corresponding $\widetilde{v} \in \mathbb{P}_1(\mathcal{T})$ defined as, $\widetilde{v}\big|_T = v\big|_T \overline{r}_T$, then we have that u_ε^h verifies also (3.24). Since we know that u_ε^h minimizes the energy functional (3.25) we have the following inequality,

$$\begin{aligned}
&\left(\nabla u_\varepsilon^h, \nabla u_\varepsilon^h\right)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} + h^{-\sigma}(u_\varepsilon^h, u_\varepsilon^h)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} - 2(f, u_\varepsilon^h)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} \\
\le &\left(\nabla v^h, \nabla v^h\right)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} + h^{-\sigma}(v^h, v^h)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} - 2(f, v^h)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)}.
\end{aligned}$$

Furthermore we notice that for any function $v \in \mathcal{W}^{1,2}(|\mathbf{x}|^{-\gamma},\Omega)$ we can consider $\mathbf{x} \mapsto v|\mathbf{x}|^\gamma \in \mathcal{W}^{1,2}(\Omega)$ and use (1.16) to obtain,

$$\left(\nabla u_0, \nabla v\right)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} + h^{-\sigma}(u_0, v)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)} = (f, v)_{\mathcal{L}^2(|\mathbf{x}|^\gamma,\Omega)}.$$

Combing the last two equations we obtain a weighted version of (3.10),

$$\begin{aligned}
&\left|u_0 - u_\varepsilon^h\right|^2_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma + u_\varepsilon^h\right)^2 |\mathbf{x}|^\gamma \, ds \\
\le &\left|u_0 - I_{\mathcal{N}}^1 u_0\right|^2_{\mathcal{W}^{1,2}(|\mathbf{x}|^\gamma,\Omega)} + h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma + I_{\mathcal{N}}^1 u_0\right)^2 |\mathbf{x}|^\gamma \, ds.
\end{aligned} \tag{3.26}$$

Theorem 4.9. *Let u_0 be the solution to* (1.16)*, u_ε^h be the solution to* (3.6)*, then the following a priori error estimate holds,*

$$\left\|u_0 - u_\varepsilon^h\right\|_{\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}},\Omega)} \le C h^\mu \|f\|_{\mathcal{L}^2(\Omega)}$$

where $\mu = \min\left\{1, \frac{\sigma}{2}\right\}$.

Proof. The proof of this result is a repetition of the steps described in Theorem 2.3 using (3.26), together with the interpolation estimates developed in the previous Chapter and the fact that

$\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ is a Muckenhoupt weighted Sobolev space and therefore we can use the usual trace inequality. Last one needs to apply the same observation made in Remark 2.5. $\square$

Conclusions

I would like now to draw the main conclusion of my thesis regarding the penalty finite element methods applied to non smooth domains. First I proved an a priori error estimate using techniques developed for singular data in [85] and a duality trick proposed in [7]. Those estimates establish the non inferiority of penalty finite element methods compared to classical conforming finite elements with respects to the error in $\mathcal{W}^{1,2}(\Omega)$ and $\mathcal{L}^2(\Omega)$, provided that the correct penalisation term is chosen. Those results are contained in Theorems 2.3, 2.6 and in Remarks 2.5 and 2.7. I then numerically observed that if we choose the penalisation term in a specific manner then the penalty finite element method converges optimally with respect to the $\mathcal{L}^2(\Omega)$ norm, see Figure 4.3. I've also showed the known result that conforming finite element methods cannot achieve optimal rate of converge. When we mention an appropriate choice of penalisation term we mean $\varepsilon = h^{5/3}$ as numerically showed in Figure 4.2 and Figure 4.3.
I later addressed the question, "Is it possible to prove by a duality trick that the penalty finite element converges optimally with respect to the $\mathcal{L}^2(\Omega)$ norm ?". In particular I found a negative answer in both the classical and weighted regularity setting. Now a very legitimate point that the reader can make is why do we care if it is possible to prove the above result by a duality argument. The answer to this question lies in the fact that the standard Babuska-Osborn theory for the approximation of eigenvalue problems yields an a priori error estimate by using a duality trick. Furthermore from Figure 4.6 we notice that the eigenvalues approximation using penalty finite elements converges with order 2 which is even better than the super-convergence we observed for the source problem in $\mathcal{L}^2(\Omega)$. More details on this topic can be found in [13].
Last, I proved that both the conforming finite element method and the penalty finite element method, with the correct penalisation term, converge with optimal error rate in the norm associated with the natural choice of Muckenhoupt weighted Sobolev space used to study the regularity in domains presenting point singularity. I redirect the reader interested in an argument regarding why this result is morally relevant to [63].
Further work will involve more detailed investigations of the super-convergence with respect to the $\mathcal{L}^2(\Omega)$ norm, presented in Figure 4.3. In particular it would be interesting to recast the penalty finite element method in the framework of energy corrected finite elements presented in [44], for which a second order convergence in $\mathcal{L}^2$ can be proven. Moreover as previously discussed, Figure 4.6 motivates us to look into a weighted Babuska-Osborn theory, this will be a delicate task since from Figure 4.5 we know that the source problem converges with order 1 in the weighted $\mathcal{W}^{1,2}(\Omega)$ norm even if the same behaviour is also presented by conforming finite elements for which the approximation of the eigenvalues does not super-converge. Last it would

be interesting to investigate what happens in the mixed Laplacian problem if we imposed the homogeneous Neumann boundary conditions using a penalisation method.

4

Appendix – Numerical Experiments

All the numerical experiments that are presented in this thesis were realised using the finite element library NGSolve [84],[83]. Furthermore all the code to reproduce the numerical experiments here presented can be found in the repository dedicated to my master thesis. The eigenvalues computation were performed using the SLEPc library [64]. I'd like to thank Stefano Zampini for teaching me about the PETSc library and how to use PETSc4py and SLEPc4py.

Fig. 4.1: The figure shows how the error of the conforming finite element method, described in chapter 3, decays when it is measured with respect to the $\mathcal{W}^{1,2}(\Omega)$ and $\mathcal{L}^2(\Omega)$ norms.

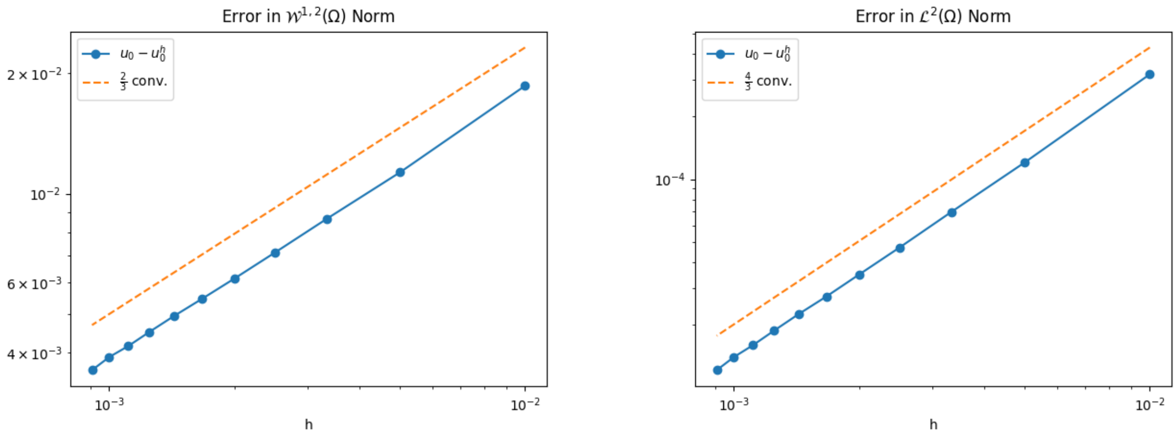


Fig. 4.2: The figure shows how the error of the penalty finite element method, described in Chapter 3, decays when measured with respect to the $\mathcal{W}^{1,2}(\Omega)$ and $\mathcal{L}^2(\Omega)$ norms varying σ.

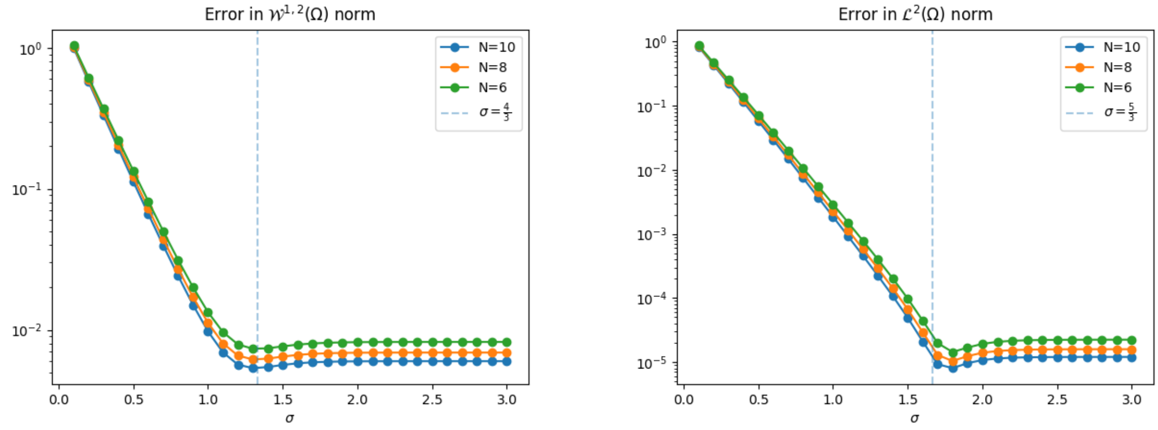


Fig. 4.3: The figure shows how the error of the penalty finite element method, described in Chapter 3, decays when it is measured with respect to the $\mathcal{L}^2(\Omega)$ norm for $\sigma = 1.2, 1.7, 1.8$ and 2.0.

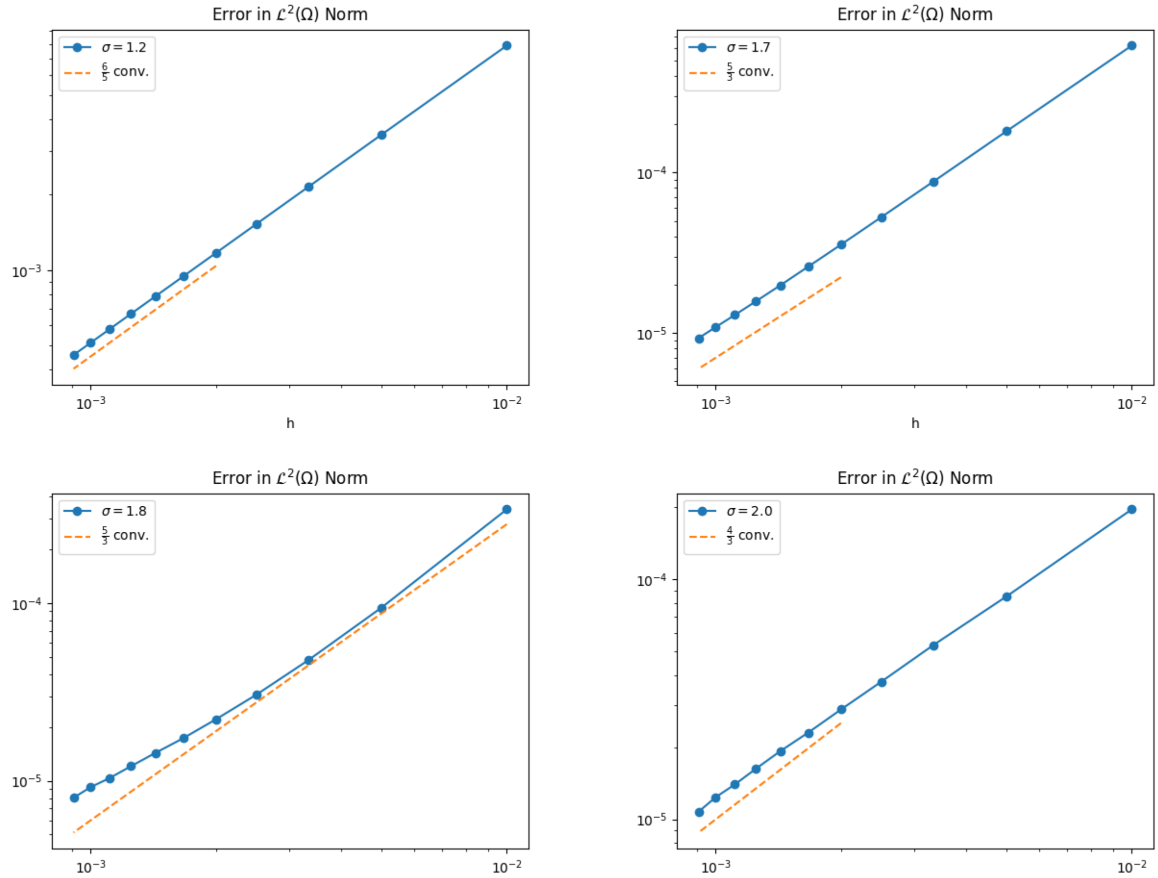


Fig. 4.4: The figure shows how the error of the conforming finite element method, described in Chapter 3, decays when it is measured with respect to the $\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ and $\mathcal{L}^2(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ norms.

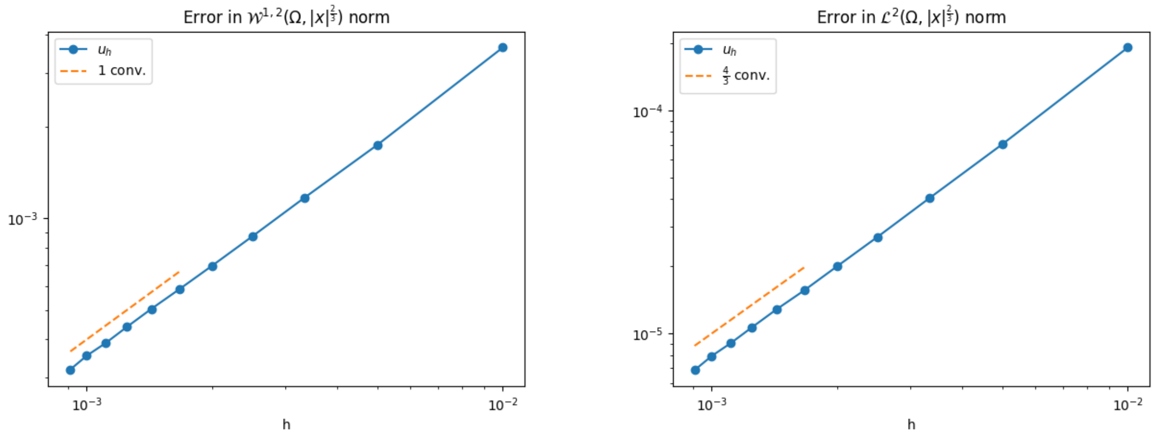


Fig. 4.5: The figure shows how the error of the penalty finite element method, described in Chapter 3, decays when it is measured with respect to the $\mathcal{W}^{1,2}(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ and $\mathcal{L}^2(|\mathbf{x}|^{\frac{2}{3}}, \Omega)$ norms.

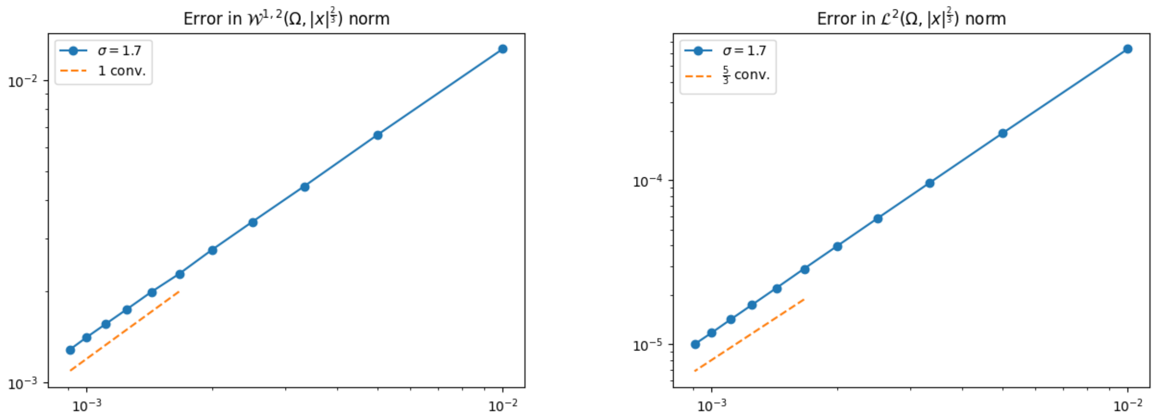
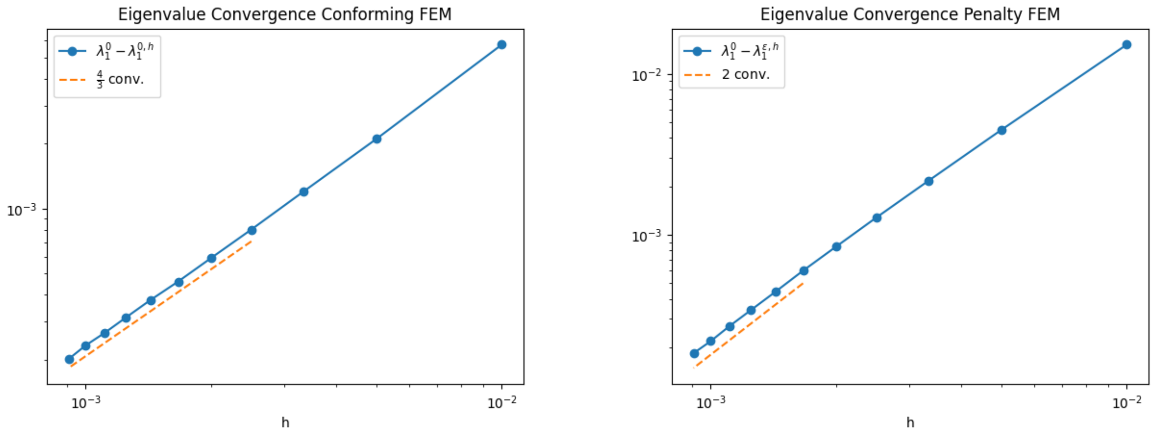


Fig. 4.6: The figure show the convergence of the eigenvalue corresponding to the first moment of the Poisson equation (on the domain depicted in Figure 1.2), for the conforming finite element method and the penalty finite element method.



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